Can compute the genus of $C \sim aF_1 + bF_2$ using adjunction $\left(= (a-1)(b-1)\right)$.

<u>Ruled surfaces:</u>

<u>Definition:</u> A ruled surface is a surface X with a surjective morphism $\pi: X \longrightarrow C$ where C is a curve (projective non singular curve) s.t. all the fibers of π are isomorphic to $\mathbb{P}^1_k$.

<u>Facts about ruled surfaces:</u> One can show that π has a section, i.e., a closed embedding $\sigma: C \hookrightarrow X$ s.t. $\pi \circ \sigma = Id_C$. Also, there exists a locally free sheaf $\mathcal{E}$ of rank 2 on C s.t. $X \cong \mathbb{P}_C(\mathcal{E})$.

We saw last quarter (homework) that any two such locally free sheaves differ by tensor product with an invertible sheaf on C.

One can prove that for any section $\sigma: C \hookrightarrow X$, if we denote C_σ the image of σ, then $\mathrm{Pic}(X) \cong \mathbb{Z}[C_\sigma] \oplus \pi^* \mathrm{Pic}(C)$ and $\mathrm{Num}(X) \cong \mathbb{Z}[C_\sigma] \oplus \mathbb{Z}[F]$ where F is any fiber of $\pi: X \twoheadrightarrow C$ $(F \cong \mathbb{P}^1_k)$.

There is a 1-to-1 correspondence between sections of π and invertible quotients of $\mathcal{E}$, sending σ to $\sigma^* \mathcal{O}_{\mathbb{P}_C(\mathcal{E})}(1)$.

There are minimal choices of locally free sheaves $\mathcal{E}$ s.t. $X \cong \mathbb{P}_C(\mathcal{E})$ and $h^0(\mathcal{E}) > 0$. This means that for all $\mathcal{L} \in \mathrm{Pic}(C)$ of negative degree, $h^0(\mathcal{E} \otimes \mathcal{L}) = 0$.

The degree of such minimal $\mathcal{E}$ is an invariant of X, usually denoted $-e$. There exists a section σ with $\mathcal{O}_X(C_\sigma) \cong \mathcal{O}_{\mathbb{P}_C(\mathcal{E})}(1)$. Such a section has self-intersection $C_\sigma^2 = -e$.

One can compute $\omega_X \cong \pi^* \omega_C \otimes \Lambda^2 \mathcal{E}(-2C_\sigma)$.

<u>Definition:</u> The arithmetic genus of a surface X is the integer $p_a(X) := \chi(\mathcal{O}_X) - 1 = h^0(\mathcal{O}_X) - h^1(\mathcal{O}_X) + h^2(\mathcal{O}_X) - 1$

$$= h^2(\mathcal{O}_X) - h^1(\mathcal{O}_X)$$

The geometric genus is $p_{g,X} := h^0(\omega_X) = h^2(\mathcal{O}_X)$.
(Serre Duality)

The irregularity of X is $q_X := h^1(\mathcal{O}_X) = h^1(\omega_X)$.

If X is a ruled surface, then $p_{g,X} = 0$, $q_X = g_C$,
$p_{a,X} = -g_C$.

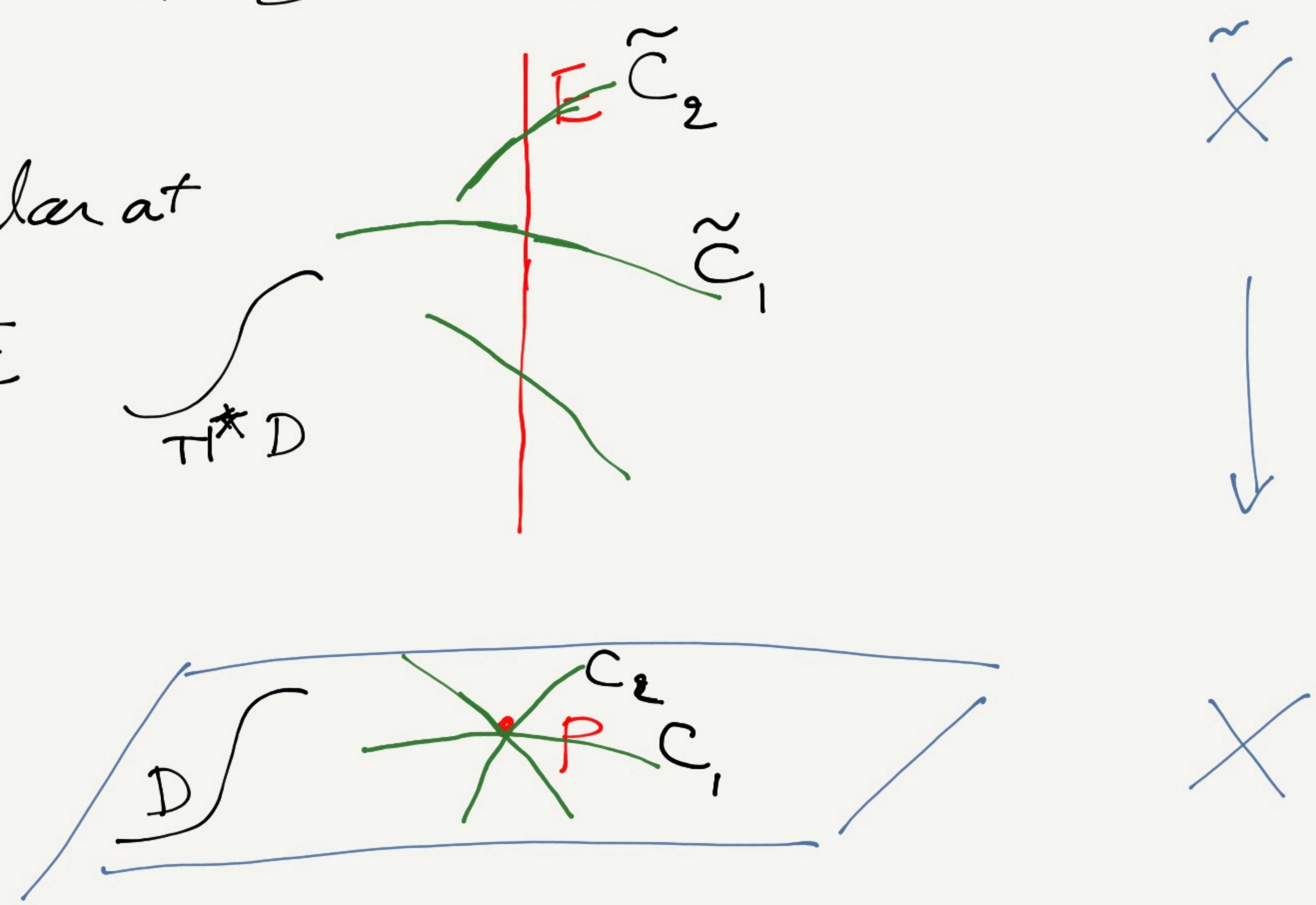
<u>Blow-ups of surfaces:</u> X surface.

$P \in X$ closed point. $\pi : \tilde{X} \to X$ the blow-up of X along P (i.e., along $\mathcal{I}_P$). Then, $\tilde{X}$ is also a nonsingular projective surface. The exceptional curve E (i.e., $\pi^{-1}(P) \subset \tilde{X}$) is isomorphic to $\mathbb{P}^1$ and has self-intersection $E^2 = -1$. Conversely, we can show that any smooth rational curve with self-intersection -1

can be blown down to a point. (i.e., it is the exceptional curve of a blowup of a point).

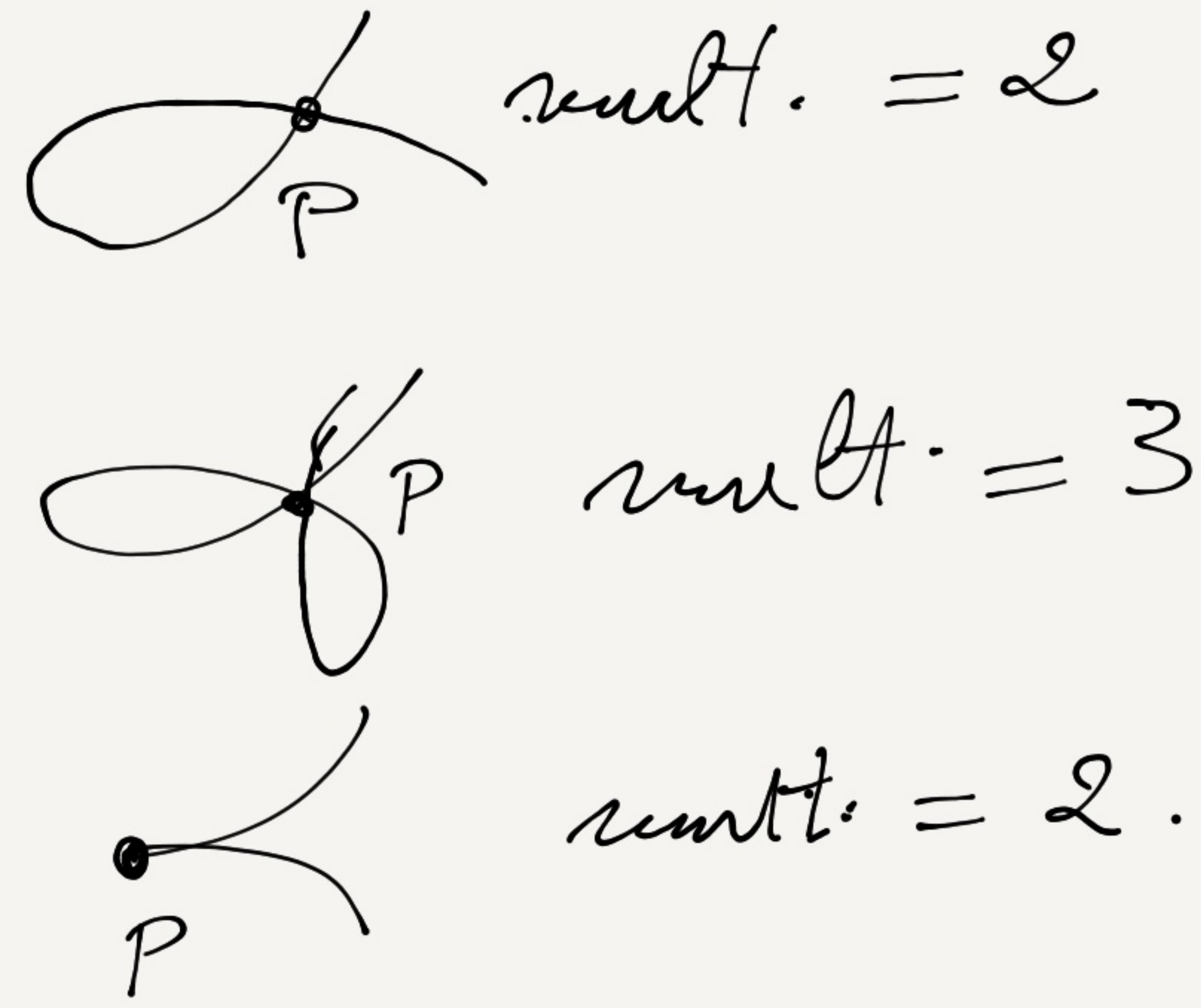
One has $\mathrm{Pic}(\tilde{X}) = \pi^* \mathrm{Pic}(X) \oplus \mathbb{Z}[E]$ and the decomposition is orthogonal with respect to the intersection pairing on $\tilde{X}$. Furthermore, $\forall$ C, D curves on X, $\quad \pi^* C \cdot \pi^* D = C \cdot D$.

If C is nonsingular at P: $\pi^* C = \tilde{C} + E$



If C has multiplicity m at P, then

$$\pi^* C = \tilde{C} + mE$$



The canonical sheaf of $\tilde{X}$

is $\quad \omega_{\tilde{X}} \cong \pi^* \omega_X(E) := \pi^* \omega_X \otimes \mathcal{O}_{\tilde{X}}(E)$

Furthermore: $\quad \pi_* \mathcal{O}_{\tilde{X}} \cong \mathcal{O}_X$ and $H^i(\mathcal{O}_{\tilde{X}}) \cong H^i(\mathcal{O}_X)$.

<u>Del Pezzo surfaces:</u> A Del Pezzo surface of degree d a blow-up of $\mathbb{P}^2$ at $9-d$ general points. $(1 \leq d \leq 8)$.

The canonical sheaf of $X = Bl_{\{P_1, \dots, P_{9-d}\}}(\mathbb{P}^2)$

P_i distinct, general

is $\omega_X \cong \pi^* \omega_{\mathbb{P}^2}(E_1 + \dots + E_{9-d})$

we know $\omega_{\mathbb{P}^2} \cong \mathcal{O}_{\mathbb{P}^2}(-3) \cong \mathcal{O}_{\mathbb{P}^2}(-3H)$ $\quad \underset{\text{line}}{H} \subset \mathbb{P}^2$

$$\omega_X = \mathcal{O}_X(K_X) = \pi^* \mathcal{O}_{\mathbb{P}^2}(-3H)(E_1 + \dots + E_{9-d})$$

$$= \mathcal{O}_X(-3\pi^* H + E_1 + \dots + E_{9-d})$$

$$K_X^2 = (-3\pi^* H + E_1 + \dots + E_{9-d})^2$$

$$= 9 \cdot H^2 - 6\pi^* H \cdot (E_1 + \dots + E_{9-d}) + (E_1 + \dots + E_{9-d})^2$$

$$= 9 - 0 + E_1^2 + \dots + E_{9-d}^2 = 9 - (9-d) = d \geq 1$$

One can show $-K_X$ is ample.

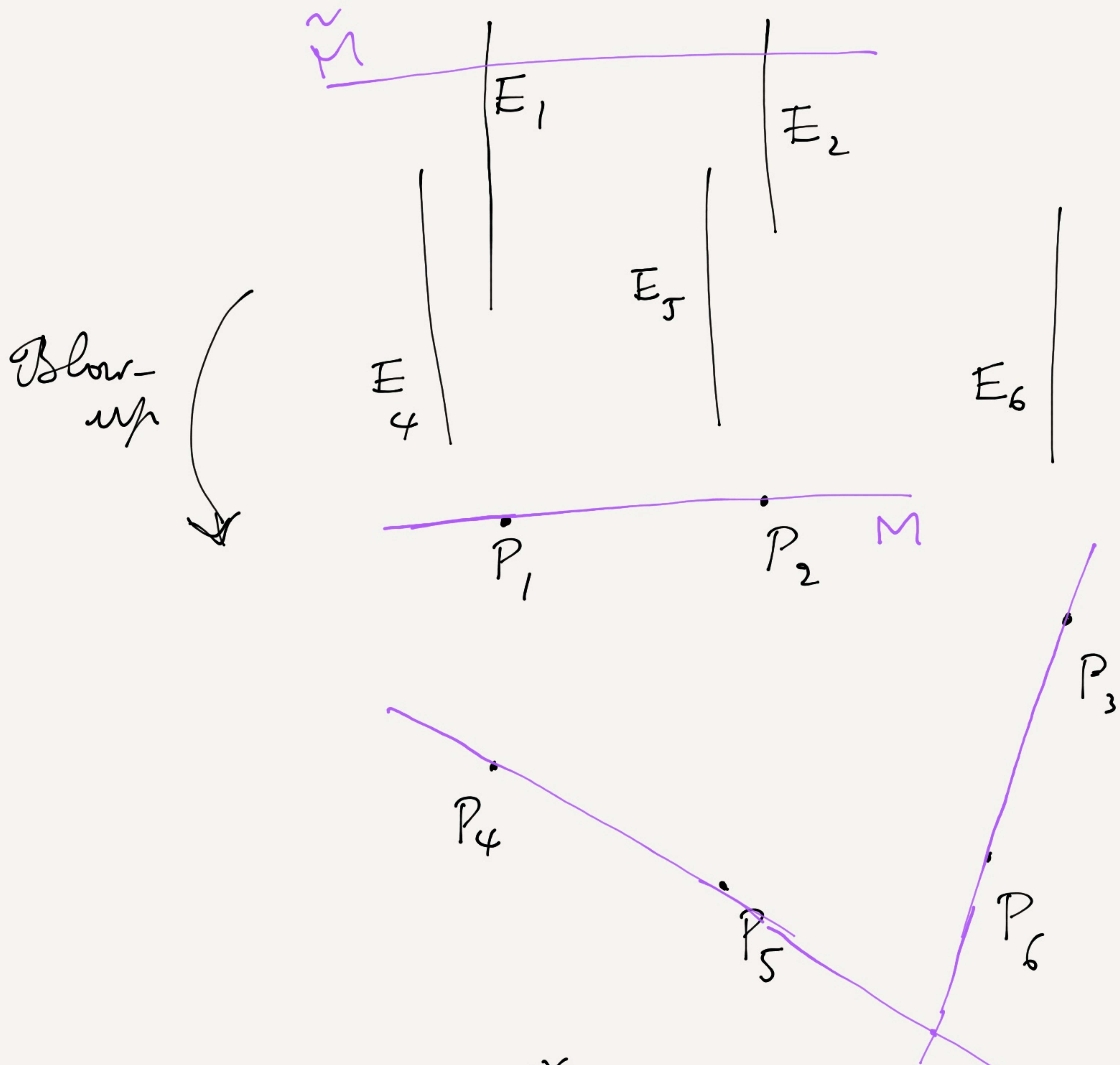
When $d \geq 3$, one can show $|-K_X|$ is in fact very ample and defines a closed embedding

$$X \hookrightarrow \mathbb{P}^d$$

where the image of X is a surface of degree d.

For $d=3$, this is a cubic surface in $\mathbb{P}^3$, i.e; the zeros of a cubic polynomial.

Each hyperplane $M \subset \mathbb{P}^d$ intersects X in a divisor of $|-K_X|$. A curve $L \subset X$ maps to a line $L \subset \mathbb{P}^d$ when $L \cdot M =$ one point, this means $L \cdot (-K_X) = 1$ or $L \cdot K_X = -1$.



$$X \hookrightarrow \mathbb{P}^3$$
$$|-K_X|$$

$$\mathbb{P}^2$$

$$\mathcal{O}_{\mathbb{P}^2}(H) = \mathcal{O}_{\mathbb{P}^2}(1)$$

$$-K_X = 3\pi^* H - E_1 - \cdots - E_6$$

curves L in X s.t. $g_L = 0$ and $L \cdot K_X = -1$

adjunction: $-2 = 2g_L - 2 = L\cdot(K_X + L) = -1 + L^2$

$$\Rightarrow L^2 = -1$$

So we look for $L \subset X$ s.t. $L^2 = -1$ and $L \cdot K_X = -1$

M_{12} = line through P_1 and P_2 in $\mathbb{P}^2$

$$\pi^* M_{12} = \tilde{M}_{12} + E_1 + E_2$$

$$1 = M_{12}^2 = (\pi^* M_{12})^2 = \tilde{M}_{12}^2 + E_1^2 + E_2^2 + 2(M_{12}\cdot E_1 + M_{12}\cdot E_2)$$

$$= \tilde{M}_{12}^2 - 1 - 1 + 4 = \tilde{M}_{12}^2 + 2$$

$$\Rightarrow \tilde{M}_{12}^2 = -1$$

$$\tilde{M}_{12}\cdot K_X = (\pi^* M_{12} - E_1 - E_2)\cdot(3\pi^* H - E_1 - \cdots - E_6)$$

$$= 3 + (E_1 + E_2)(E_1 + \cdots + E_6) = 3 + 2(E_1^2 + E_2^2)$$

$$= 3 - 4 = -1$$

For the E_i : $E_i^2 = -1$

$$E_i \cdot K_X = E_i \cdot (-3\pi^* H + E_1 + \cdots + E_6)$$
$$= E_i^2 = -1$$

Conics through 5 of the P_i's: $C_i :=$ conic through $E_1, \ldots, \hat{E}_i, \ldots, E_6$

similar computation $\Rightarrow$ $\tilde{C}_i^2 = -1 = \tilde{C}_i \cdot K_X$

One can show these give all the lines:

6 exceptional curves, 6 proper transforms $\tilde{C}_i$ of conics.

$15 = \binom{6}{2}$ lines M_{ij} through $P_i \neq P_j$.

$\Rightarrow$ 27 lines in the cubic surface.