MATH 202A
APPLIED ALGEBRA I
FALL 2021

Homework week 1

Due by 2359 on Sunday October 3 (hand in via Gradescope).

1. Prove, using only the results from Section 1.2, that (i) any finitely-generated vector space has a (finite) basis, and (ii) any two bases of a finitely-generated vector space V have the same size.

[Recall that we say a vector spaces is finitely-generated to mean that it has a finite spanning set.]

2. Suppose n is a positive integer and a collection of subsets $S_1, \dots, S_k \subseteq \{1, \dots, n\}$ (not necessarily distinct) are given. A set $I \subseteq \{1, \dots, k\}$ is called *awesome* if it has the following property: for every $x \in \{1, \dots, n\}$, the number

$$|\{i \in I : x \in S_i\}|$$

is even.

Prove that the number of awesome sets $I \subseteq \{1, \dots, k\}$ is always a power of 2.

3. Let V, W be vector spaces, $U \subseteq V$ a subspace, and $\phi \colon V \to W$ a linear map. Suppose also that $\ker \phi \supseteq U$.

Prove that there exists a unique linear map $\psi \colon V/U \to W$ such that $\phi(v) = \psi(v+U)$ for all $v \in V$.

Prove that ψ is injective if and only if $U = \ker \phi$.

4. Let V be a finite-dimensional vector space and $U \subseteq V$ a subspace. Suppose that $u_1, \dots, u_k$ are a basis for U, and $v_1, \dots, v_m$ are vectors in V such that $v_1 + U, \dots, v_m + U$ is a basis for V/U.

Prove that $u_1, \dots, u_k, v_1, \dots, v_m$ is a basis for V (and hence $\dim V = \dim U + \dim(V/U)$).

5. Suppose V is a vector space over $\mathbb{R}$, that $w_1, w_2, w_3 \in V$ is a spanning set for V, and that u_1, u_2, u_3 are linearly independent where

$$\begin{aligned} u_1 &= 4w_1 - 3w_3 \\ u_2 &= -w_1 + w_3 \\ u_3 &= w_1 + w_2 + w_3. \end{aligned}$$

It is a consequence of Theorem 1.2.1 that u_1, u_2, u_3 spans V (since we can add $3 - 3 = 0$ elements from w_1, w_2, w_3 to make it span). In particular, w_1, w_2, w_3 lie in $\operatorname{span}(u_1, u_2, u_3)$, meaning there exist coefficients $a_{ij} \in \mathbb{R}$ such that $w_j = a_{j1} u_1 + a_{j2} u_2 + a_{j3} u_3$.

Our proofs are constructive: that is, we can—and will—find the coefficients a_{ij} explicitly by close inspection of *the proof of Theorem 1.2.1* and its subsidiary lemmas.

This "expanded" proof is as follows. Set $B_0 = w_1, w_2, w_3$.

Step 1: Consider the spanning list u_1, w_1, w_2, w_3. It is linearly dependent (by Lemma 1.2.3, as B_0 spans). Using this dependence relation and Lemma 1.2.2, we discover that $w_3 \in \operatorname{span}(u_1, w_1, w_2)$ and $B_1 = u_1, w_1, w_2$ spans V.

Step 2: Consider the spanning list u_2, u_1, w_1, w_2. It is linearly dependent (by Lemma 1.2.3, as B_1 spans). Using this dependence relation and Lemma 1.2.2, we discover that $w_1 \in \operatorname{span}(u_2, u_1)$ and $B_2 = u_2, u_1, w_2$ spans V.

Step 3: Consider the spanning list u_3, u_2, u_1, w_2. It is linearly dependent (by Lemma 1.2.3, as B_2 spans). Using this dependence relation and Lemma 1.2.2, we discover that $w_2 \in \operatorname{span}(u_3, u_2, u_1)$ and $B_3 = u_3, u_2, u_1$ spans V.

Now, in each Step i ($1 \le i \le 3$):—

(a) find an explicit dependence relation in the length 4 list, and explicit coefficients in each of the statements after "we discover that" (i.e., respectively $w_3 \in \operatorname{span}(u_1, w_1, w_2)$, $w_1 \in \operatorname{span}(u_2, u_1)$, $w_2 \in \operatorname{span}(u_3, u_2, u_1)$);

(b) using your answers to (a), find explicit coefficients for each of the statements $u_1 \in \operatorname{span}(B_i)$, $u_2 \in \operatorname{span}(B_i)$, $u_3 \in \operatorname{span}(B_i)$, and summarize your answers in a 3×3 matrix;

(c) using your answers to (a), find explicit coefficients for each of the statements $w_1 \in \operatorname{span}(B_i)$, $w_2 \in \operatorname{span}(B_i)$, $w_3 \in \operatorname{span}(B_i)$, and summarize your answers in a 3×3 matrix.

Note the similarity of this process with any other algorithms (Gaussian elimination / reducing to RREF / LU decomposition / etc.) with which you are familiar. [You need not write anything for this part.]