MATH 202A
APPLIED ALGEBRA I
FALL 2021

Homework week 2

Due by 2359 on Sunday October 10 (hand in via Gradescope).

1. For each of the following, determine whether it is an inner product on $\mathbb{R}^2$.

(a) $\langle (x,y),(x',y')\rangle = xx' - yy'$.

(b) $\langle (x,y),(x',y')\rangle = xx' + xy' + yx' + 2yy'$.

(c) $\langle (x,y),(x',y')\rangle = xx' + xy' + yx' + yy'$.

2. Let V be a finite dimensional inner product space, with inner product $\langle -,-\rangle$ as usual.

(a) Suppose $\phi\colon V \to V$ a linear map. Define a new operation $\langle -,-\rangle_1 \colon V \times V \to F$ by

$$\langle v,w\rangle_1 = \langle \phi(v),\phi(w)\rangle \ .$$

Show that if ϕ is invertible then this is an inner product on V.

(b) In the set-up of (a), show that if ϕ is not invertible then $\langle -,-\rangle_1$ is not an inner product.

(c) Conversely, suppose $\langle -,-\rangle_2$ is yet another inner product on V. Show that there is an invertible linear map $\psi\colon V \to V$ such that

$$\langle v,w\rangle_2 = \langle \psi(v),\psi(w)\rangle \ .$$

3. Consider the vectors

$$\begin{aligned} v_1 &= (1.1, 1.1, 1.1, 1.1)\\ v_2 &= (3.3, 3.3, 1.1, 1.1)\\ v_3 &= (6.6, 4.4, 2.2, 0)\\ v_4 &= (10, 5.4, 3.2, -1.0)\end{aligned}$$

in $\mathbb{R}^4$, which carries the usual dot product.

You may assume that

$$\begin{pmatrix} 1.1 & 3.3 & 6.6 & 10\\ 1.1 & 3.3 & 4.4 & 5.4\\ 1.1 & 1.1 & 2.2 & 3.2\\ 1.1 & 1.1 & 0 & -1.0\end{pmatrix} = \begin{pmatrix} 0.5 & 0.5 & 0.5 & 0.5\\ 0.5 & 0.5 & -0.5 & -0.5\\ 0.5 & -0.5 & 0.5 & -0.5\\ 0.5 & -0.5 & -0.5 & 0.5\end{pmatrix}\begin{pmatrix} 2.2 & 4.4 & 6.6 & 8.8\\ 0 & 2.2 & 4.4 & 6.6\\ 0 & 0 & 2.2 & 4.4\\ 0 & 0 & 0 & 0.2\end{pmatrix}.$$

Show that there exist coefficients $a_1, a_2, a_3 \in \mathbb{R}$ such that

$$\|v_4 - a_1 v_1 - a_2 v_2 - a_3 v_3\| \leq 0.2.$$

[Please tell me if you spot a mistake in these numbers.]

4. Let $\mathcal{P}_{\leq 4}$ denote the vector space of real-valued polynomials of degree ≤ 4:

$$\mathcal{P}_{\leq 4} = \{p(X) = a_0 + a_1 X + \cdots + a_4 X^4 \colon a_0, \ldots, a_4 \in \mathbb{R}\}.$$

You may use without proof that $B = 1, X, X^2, \ldots, X^4$ and $B' = 1, 1+X, (1+X)^2, \ldots, (1+X)^4$ are two basis for $\mathcal{P}_{\leq 4}$, and that

$$\phi \colon p(X) \mapsto \frac{dp}{dX}$$

and

$$\psi \colon p \mapsto (X \mapsto p(X+1))$$

are linear maps $\mathcal{P}_{\leq 4} \to \mathcal{P}_{\leq 4}$. (So e.g. $\psi(3+4X) = 3 + 4(X+1) = 7 + 4X$.)

Write down:

(a) $\mathcal{M}(\phi, B, B)$;

(b) $\mathcal{M}(\psi, B, B)$;

(c) $\mathcal{M}(\mathrm{id}, B', B)$;

(d) $\mathcal{M}(\mathrm{id}, B, B')$;

(e) $\mathcal{M}(\phi, B, B')$.

For a subspace $U \subseteq V$, write

$$U^\perp = \{\phi \in V^* \colon \phi(u) = 0 \ \forall u \in U\} \subseteq V^*.$$

Similarly, if $W \subseteq V^*$, write

$$W^\perp = \{u \in V \colon \phi(u) = 0 \ \forall \phi \in W\} \subseteq V.$$

5. Prove that if V is finite-dimensional (and so we can identify V with V^{**}) then $(U^\perp)^\perp = U$.

6. Let V and W be vector spaces, let $f \colon V \to W$ be a linear map, and let $f^* \colon W^* \to V^*$ be the dual map.

(a) Prove that $\ker(f^*) = (\operatorname{im} f)^\perp$.

Now assume that V, W are finite-dimensional (and so we can identify V with V^{**} and W with W^{**}).

(b) Prove that $f^{**} = f$.

(c) Prove that $(\operatorname{im}(f^*))^\perp = \ker f$.

(d) Prove that $(\ker(f^*))^\perp = \operatorname{im} f$.

(e) Prove that $\operatorname{im}(f^*) = (\ker f)^\perp$.