MATH 202A
APPLIED ALGEBRA I
FALL 2021

Homework week 3

Due by 2359 on Sunday October 17 (hand in via Gradescope).

1. Suppose V, W are a finite dimensional inner product spaces and $\phi\colon V \to W$ is a linear map. Show that if ϕ is injective then $\phi^* \circ \phi$ is injective (and hence an isomorphism).

2. Prove that if V is a finite-dimensional inner product space, $\phi\colon V \to V$ is a normal operator and $\lambda, \mu \in F$ are scalars with $\lambda \neq \mu$ then $E(\lambda, \phi) \perp E(\mu, \phi)$ (i.e. these subspaces are orthogonal).

3. Suppose V is a finite-dimensional vector space, $\phi\colon V \to V$ is a diagonalizable linear map, and $U \subseteq V$ is a ϕ-invariant subspace.

Prove that the restriction $\phi|_U\colon U \to U$ is diagonalizable.

[**Hint**: the maps Φ_i from the proof of Proposition 3.1.7 might come in handy here.]

4. Suppose V is a finite-dimensional vector space and $\phi, \psi\colon V \to V$ are two linear maps such that $\phi \circ \psi = \psi \circ \phi$ (i.e., the maps commute).

(a) For any $\lambda \in F$, prove that $E(\lambda, \phi)$ is ψ-invariant.

(b) Suppose that ϕ, ψ are diagonalizable. Prove that there is a basis B for V consisting of simultaneous eigenvectors of ϕ, ψ.

[**Hint**: remember Q3.]

5. Let V be a finite-dimensional inner product space, $\phi\colon V \to V$ a linear map and suppose $U \subseteq V$ is ϕ-invariant. Write $\psi\colon U \to U$ for the restriction $\phi|_U$.

(a) Give an example to show that ψ^* need not be the same thing as the restriction $\phi^*|_U$.

(b) Prove that, however, $\psi^* = P_U \circ (\phi^*|_U)$.

(c) Prove that if U is ϕ^*-invariant then $\psi^* = \phi^*|_U$.

6. Let V, W be finite-dimensional inner product spaces, and let $\phi\colon V \to W$ and $\psi\colon V \to V$ be linear maps.

(a) Prove that, for any $\lambda \in F$, $\dim E(\lambda, \psi) = \dim E(\overline{\lambda}, \psi^*)$.

(b) Prove that, for any $\lambda \in F$, $\lambda \neq 0$, $\dim E(\lambda, \phi^* \circ \phi) = \dim E(\lambda, \phi \circ \phi^*)$.

(c) Determine the relationship between $\dim E(0, \phi^* \circ \phi)$ and $\dim E(0, \phi \circ \phi^*)$.

7. My office has three chairs, arranged in a row. Every hour, I roll a 4-sided die and move chairs, or not, according to the following rule:

- if I'm in an end chair, I stay put if I roll a 1, 2 or 3 and move to the middle if I roll a 4;

- if I'm the middle chair, I move left on a 1 roll, stay put on a 2 or 3 roll and move right on a 4 roll.

So, if my probabilities of being in each chair one hour are (p_L, p_M, p_R), then the probabilities for the next hour are given by

$$\phi(p_L, p_M, p_R) = \left(\frac{3}{4}p_L + \frac{1}{4}p_M, \frac{1}{4}p_L + \frac{1}{2}p_M + \frac{1}{4}p_R, \frac{1}{4}p_M + \frac{3}{4}p_R\right) .$$

I start in the left chair. Show that the probability I am in the left chair again after 1000 straight hours in my office is $1/3 + \varepsilon$ where $|\varepsilon| \leq 2 \cdot (3/4)^{1000}$ (which is around $2.3 \cdot 10^{-125}$).