MATH 202A
APPLIED ALGEBRA I
FALL 2021

Homework week 4

Due by 23:59 on Sunday 24th October (hand in via Gradescope).

1. Let V be a finite-dimensional inner product space and $\phi\colon V \to V$ a self-adjoint linear map.

(a) Suppose ϕ is non-negative definite, and $v \in V$, $v \neq 0$ satisfies $\langle \phi(v), v\rangle = 0$. Prove that ϕ is singular (i.e., not injective / not surjective / not an isomorphism).

(b) Does (a) hold if ϕ is not assumed to be non-negative definite (but is still self-adjoint)? Give either a proof or a counter-example.

2. Let V be a finite-dimensional complex vector space and $\phi\colon V \to V$ a linear map. Prove that the following are equivalent: (i) ϕ is diagonalizable and every eigenvalue λ of ϕ satisfies $|\lambda| = 1$; and (ii) there exists an inner product $\langle -, -\rangle$ on V such that ϕ is unitary.

[If you can pronounce it, you can say such a ϕ is *unitarizable*.]

3. Suppose V, W are inner product spaces and $\phi\colon V \to W$ is a linear map. (You may *not* assume V, W are finite-dimensional.)

Suppose there exists an adjoint $\phi^*\colon W \to V$ for ϕ; i.e., ϕ^* is a linear map such that $\langle \phi(v), w\rangle = \langle v, \phi^*(w)\rangle$ holds for all $v \in V$ and $w \in W$.

Let $A > 0$ be a given real number, and suppose that $\|\phi(v)\| \leq A\|v\|$ holds for all $v \in V$.

Prove that $\|\phi^*(w)\| \leq A\|w\|$ holds for all $w \in W$.

4. Let V be a finite-dimensional inner product space an $\phi\colon V \to V$ a linear map.

(a) Prove that there is a unique choice of self-adjoint linear maps $\sigma, \tau\colon V \to V$ such that $\phi = \sigma + i\tau$.

(b) Prove that $\mathcal{F}(\sigma) = \{\mathrm{Re}(z)\colon z \in \mathcal{F}(\phi)\}$.

(c) Let $\lambda_1 \geq \cdots \geq \lambda_n$ be the eigenvalues of σ with multiplicity, and similarly let $\mu_1 \geq \cdots \geq \mu_n$ be the eigenvalues of τ. Prove that

$$\mathcal{F}(\phi) \subseteq \big\{a + ib \in \mathbb{C}\colon \lambda_n \leq a \leq \lambda_1,\ \mu_n \leq b \leq \mu_1\big\}.$$

5. Let V be a finite-dimensional inner product space, let $\phi\colon V \to V$ be a self-adjoint linear map, and let $U \subseteq V$ be a subspace of codimension 1 (i.e., dimension $\dim V - 1$).

Let $\psi\colon U \to U$ be the linear map $P_U \circ \phi|_U$.

(a) Prove that ψ is self-adjoint.

(b) If $\lambda_1 \geq \cdots \geq \lambda_n$ are the eigenvalues of ϕ (with multiplicity), and $\mu_1 \geq \cdots \geq \mu_{n-1}$ are the eigenvalues of ψ (with multiplicity), prove that for each i, $1 \leq i \leq n-1$, we have $\lambda_i \geq \mu_i \geq \lambda_{i+1}$.

6. Suppose a group of 100 people, named $x_1, \ldots, x_{100}$, are such that each has exactly 50 friends in the group. Let A be the 100×100 matrix with entries $A_{ij} = 1$ if $i \neq j$ and x_i, x_j are friends, and $A_{ij} = 0$ otherwise. You may assume that friendship is symmetric, and therefore so is A. Suppose $\lambda_1 \geq \cdots \geq \lambda_{100}$ are the eigenvalues of A, with multiplicity.

(a) Prove that $\lambda_1 = 50$.

(b) Suppose further that $\lambda_2 = 40$ and $\lambda_{100} = -40$. Prove that, for any set $S \subseteq \{x_1, \ldots, x_{100}\}$ of size $|S| = 50$, the number M of pairs $xy \in S$ such that x, y are friends, satisfies

$$125 \leq M \leq 1125.$$

[**Hint:** consider the vector $y = (y_1, \ldots, y_{100})$ where $y_i = 1$ if $i \in S$ and $y_i = 0$ otherwise, and investigate the value $\langle Ay, y \rangle$.]