MATH 202A
APPLIED ALGEBRA I
FALL 2021

Homework week 5

Due by 2359 on Sunday 31st October ☺ (hand in via Gradescope).

Given an linear map ϕ between finite dimensional inner product spaces, we write $\sigma_1(\phi)$ for the largest singular value of ϕ.

1. If V is a finite-dimensional inner product space and $\phi\colon V \to V$ is a normal map with eigenvalues $\lambda_1, \ldots, \lambda_n$. Prove that the singular values of ϕ are $|\lambda_1|, \ldots, |\lambda_n|$.

2. Let U, V, W be finite-dimensional inner product spaces and $\phi\colon U \to V$, $\psi\colon V \to W$ linear maps, show that $\sigma_1(\psi \circ \phi) \le \sigma_1(\psi)\sigma_1(\phi)$.

3.(a) Let V be a finite dimensional inner product space and $\phi\colon V \to V$ a linear map. For any positive integer k, prove that $\sigma_1(\phi^k) \le \sigma_1(\phi)^k$. Prove that if ϕ is normal then $\sigma_1(\phi^k) = \sigma_1(\phi)^k$.

(b) Consider $V = \mathbb{C}^2$ and $\phi(x, y) = (x + (3/2)y, y)$. Prove that $\sigma_1(\phi)^{1000} = 2^{1000}$ but $\sigma_1(\phi^{1000}) \le 1500 + \frac{1}{1500}$.

4. My office has three chairs, arranged in a row.[1] Every hour, I receive an instruction by email from the university authorities. If the instruction says "FLIP", I roll a fair 2-sided die and accordingly move to one of the chairs I am not currently sitting in, with probability 1/2 each.

If the instruction says "FLOP", I roll a fair 4-sided die and move chairs, or not, based on the rule given in Pset 3 Q7.

So, if my probabilities of being in each chair one hour are (p_L, p_M, p_R), then the probabilities for the next hour are given by

$$\psi(p_L, p_M, p_R) = \left(\frac{1}{2}p_M + \frac{1}{2}p_R, \frac{1}{2}p_L + \frac{1}{2}p_R, \frac{1}{2}p_L + \frac{1}{2}p_M\right)$$

if the instruction is "FLIP", and

$$\phi(p_L, p_M, p_R) = \left(\frac{3}{4}p_L + \frac{1}{4}p_M, \frac{1}{4}p_L + \frac{1}{2}p_M + \frac{1}{4}p_R, \frac{1}{4}p_M + \frac{3}{4}p_R\right)$$

if the instruction is "FLOP".

The university authorities are a law unto themselves, and their instructions are unpredictable, arbitrary and follow no discernible pattern, deterministic or random. In particular you may *not* assume that FLIP and FLOP occur according to any particular random rule.

I start in the left chair. Show that the probability I am in the left chair again after 1000 straight hours in my office is $1/3 + \varepsilon$ where $|\varepsilon| \le 2 \cdot (3/4)^{1000}$ (which is around $2.3 \cdot 10^{-125}$).

[**Hint:** it might help to consider Q1, Q2 and the subspace $\{1, 1, 1\}^{\perp}$.]

[1] The chairs in my office are not actually arranged in a row; there is not enough space. For the purposes of this question, suppose hypothetically that they are.