**MATH 202A**
**APPLIED ALGEBRA I**
**FALL 2021**

Homework week 6

Due by 2359 on Sunday 7th November (hand in via Gradescope).

**1.** Let $V$ be a finite-dimensional vector space over a field $F$ and $\phi \colon V \to V$ a linear map.

**(a)** Prove that if $\phi$ has an eigenvalue $\lambda \neq 0$ then $\phi$ is not nilpotent.

**(b)** Prove that, if $F = \mathbb{C}$ and $\operatorname{spec}(\phi) = \{0\}$ then $\phi$ is nilpotent. [For this question, do not use Jordan Normal Form itself, though you may use facts that go into it.]

**(c)** If $F = \mathbb{R}$ and $V = \mathbb{R}^3$, find an example of a map $\phi$ such that $\operatorname{spec}(\phi) = \{0\}$ but $\phi$ is not nilpotent.

**2.** Let $V$ be a finite-dimensional vector space and $\phi \colon V \to V$ a linear map.

Consider the subspaces

$$\{0\} \subseteq \ker \phi \subseteq \ker(\phi^2) \subseteq \dots$$

**(a)** Prove that if $\ker(\phi^k) = \ker(\phi^{k+1})$, then in fact $\ker \phi^\ell = \ker \phi^{\ell+1}$ for all $\ell \geq k$.

**(b)** Suppose $\phi$ is nilpotent. Prove that $\phi^r = 0$ holds for some $r \leq \dim V$. [Do not use the structure theorem of nilpotent maps to answer this part.]

**3.** Let $V$ be a finite dimensional vector space and $\phi \colon V \to V$ a linear map. Suppose $v \in V$, $v \neq 0$, and suppose that for some non-negative integer $m \geq 0$ we have

$$\phi^m(v) \neq 0$$

but

$$\phi^{m+1}(v) = 0 \ .$$

Show that $v, \phi(v), \dots, \phi^m(v)$ are linearly independent.

**4.** Suppose $V$ is a finite-dimensional vector space and $\phi \colon V \to V$ is nilpotent. Suppose $n = \dim V$ and $n_1, \dots, n_m \geq 1$ are integers such that $n_1 + \cdots + n_m = n$ and there is a basis $B = (v_{i,j})_{i \in [m], j \in [n_i]}$ for $V$ such that

$$\phi(v_{i,j}) = \begin{cases} v_{i,j-1} & : j > 1 \\ 0 & : j = 1 \end{cases}$$

as guaranteed by the structure theorem for nilpotent maps.

**(a)** For $k \geq 1$, prove that

$$\dim \ker(\phi^k) = \sum_{i=1}^{m} \min(k, n_i).$$

**(b)** Hence, or otherwise, show that for $\ell \geq 1$:

$$\left|\left\{i \colon 1 \leq i \leq m,\ n_i = \ell\right\}\right| = 2 \dim \ker(\phi^\ell) - \dim \ker(\phi^{\ell+1}) - \dim \ker(\phi^{\ell-1})$$

where by convention $\phi^0 = \mathrm{id}_V$.

[This establishes uniqueness in the structure theorem for nilpotent maps: the number of blocks of each size is determined by values that clearly only depend on $\phi$. If you like, use this to deduce uniqueness for the number and sizes of blocks in Jordan Normal Form.]

**5.(a)** Let $V$ be finite dimensional and let $\psi\colon V \to V$ be a nilpotent map, where $\psi^k = 0$. Write $\phi = \psi + \lambda\,\mathrm{id}$ for some $\lambda \in F$. Show that for any integer $m \geq 0$,

$$\phi^m = \sum_{r=0}^{\min(k-1,m)} \lambda^{m-r}\binom{m}{r}\psi^r.$$

**(b)** For any integer $m \geq 0$, compute the matrix $J_k^m$. Using (a), or otherwise, also compute $J(k,\lambda)^m$. Here

$$J_k = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & & & & & \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}, \quad J(k,\lambda) = \begin{pmatrix} \lambda & 1 & 0 & \cdots & 0 & 0 \\ 0 & \lambda & 1 & \cdots & 0 & 0 \\ \vdots & & & & & \\ 0 & 0 & 0 & \cdots & \lambda & 1 \\ 0 & 0 & 0 & \cdots & 0 & \lambda \end{pmatrix}$$

are $k \times k$ matrices.

**6.** Consider the vector space of infinite complex sequences $(a_0, a_1, a_2, \dots)$. Let $V$ denote the subspace that satisfy the homogeneous recursion relation

$$a_n = 6a_{n-1} - 12a_{n-2} + 8a_{n-3}$$

for all $n \geq 3$. [You need not verify that this is a subspace.] Also let $\phi\colon V \to V$ denote the "infinite left shift" map

$$\phi(a_0, a_1, a_2, \dots) = (a_1, a_2, a_3, \dots).$$

You should satisfy yourself that this latter sequence is indeed in $V$, but do not need to write anything.

**(a)** Prove that the linear map $\psi\colon V \to \mathbb{C}^3$, $\psi(a_0, a_1, a_2, \dots) = (a_0, a_1, a_2)$ is an isomorphism, and write down a matrix for $\phi$ with respect to the basis $B = \psi^{-1}(e_1), \psi^{-1}(e_2), \psi^{-1}(e_3)$ where $e_1, e_2, e_3$ is the standard basis for $\mathbb{C}^3$.

**(b)** Hence, or otherwise, compute $\mathrm{spec}(\phi)$, and deduce that $\phi - 2\,\mathrm{id}_V$ is nilpotent.

**(c)** Using Q4(i), or otherwise, and noting that $a_n$ is the first entry of $\phi^n(a_0, a_1, \dots)$, find a closed-form formula for $a_n$ in terms of $a_0, a_1, a_2$.