MATH 202A
APPLIED ALGEBRA I
FALL 2019

Homework week 7

Due by 2359 on Sunday November 14 (hand in via Gradescope).

1. Consider the space V of complex sequences $(a_0, a_1, a_2, \dots)$ satisfying

$$a_n = r_1 a_{n-1} + r_2 a_{n-2} + \cdots + r_d a_{n-d}$$

for all $n \geq d$, where $d \geq 1$ is an integer and $r_1, \dots, r_d \in \mathbb{C}$ are fixed constants. We assume also that $r_d \neq 0$. You may also assume without proof that V is finite-dimensional and $\dim V = d$.

As in Problem Set 6 Q6, write $\phi \colon V \to V$ for the linear map

$$\phi(a_0, a_1, a_2, \dots) = (a_1, a_2, a_3, \dots)$$

and also write $\psi \colon V \to \mathbb{C}$ for the linear functional $\psi(v_0, v_1, v_2, \dots) = v_0$. You may use without proof that $r_d \neq 0 \Rightarrow 0 \notin \operatorname{spec}(\phi)$.

(a) Suppose $v = (v_0, v_1, v_2, \dots) \in G(\lambda, \phi)$. By considering the action of ϕ^m, or otherwise, prove that there exists constants $c_0, \dots, c_{k-1}$ where $k = \dim G(\lambda, \phi)$, such that

$$v_m = \psi(\phi^m(v)) = \sum_{i=0}^{k-1} c_i \binom{m}{i} \lambda^m$$

for all $m \geq 0$. (By convention, $\binom{n}{i} = 0$ if $i > n$.)

(b) Prove that the sequences $v^{\lambda,i} \in V$ for $\lambda \in \operatorname{spec}(\phi)$ and $0 \leq i < \dim G(\lambda, \phi)$, given by

$$v_n^{\lambda,i} = \binom{n}{i} \lambda^n,$$

form a basis for V.

[You should be careful to explain why these sequences are in V and are linearly independent, although you should not need to prove these things directly.]

2. (a) Suppose V is an inner product space with inner product $\langle -, - \rangle$ and associated norm $\| \cdot \|$. Prove that the *parallelogram law* holds for $\| \cdot \|$: that is, for all $v, w \in V$ we have

$$\|v + w\|^2 + \|v - w\|^2 = 2\|v\|^2 + 2\|w\|^2.$$

(b) Suppose V is a finite-dimensional real vector space and $\| \cdot \|$ is a norm on V obeying the parallelogram law: that is,

$$\|v + w\|^2 + \|v - w\|^2 = 2\|v\|^2 + 2\|w\|^2$$

for all $v, w \in V$. Prove that there is an inner product $\langle -, - \rangle$ on V such that $\|v\| = \sqrt{\langle v, v \rangle}$.

[**Hint**: consider the quantity $\frac{1}{2}\left(\|v + w\|^2 - \|v\|^2 - \|w\|^2\right)$.]

[**Hint**: it may be useful to prove the identity

$$\begin{aligned} &\|v + w\|^2 + \|v - w\|^2 - \|v' + w\|^2 - \|v' - w\|^2 \\ &- \|v + w'\|^2 - \|v - w'\|^2 + \|v' + w'\|^2 + \|v' - w'\|^2 = 0 \end{aligned}$$

by applying the parallelogram law four times to (v, w), (v', w), (v, w') and (v', w'). This identity, with a suitable substitution of variables, should be what you need to prove linearity.]

(c) For every $n \geq 2$ and $1 \leq p \leq \infty$, $p \neq 2$, find vectors $v, w \in \mathbb{R}^n$ such that

$$\|v + w\|_p^2 + \|v - w\|_p^2 \neq 2\|v\|_p^2 + 2\|w\|_p^2.$$

[Hence, the $\|\cdot\|_p$-norms do not secretly come from some exotic inner product.]

(d*) Prove the same statement as (b) assuming now that V is a complex vector space.

[**Hint**: it may now help to build an expression from the quantities $\|v \pm w\|^2$ and $\|v \pm iw\|^2$.]

3. Let $V = \mathbb{R}^n$, and identify $V^* = \mathbb{R}^n$ in the usual way; i.e., a vector $(a_1, \ldots, a_n) \in \mathbb{R}^n$ corresponds to a linear map $\mathbb{R}^n \to \mathbb{R}$,

$$(x_1, \ldots, x_n) \mapsto a_1 x_1 + \cdots + a_n x_n.$$

For each of the norms $\|\cdot\|_\infty$ and $\|\cdot\|_1$ on $\mathbb{R}^n$, determine carefully the value of the corresponding dual norms $\|(a_1, \ldots, a_n)\|_\infty^*$, $\|(a_1, \ldots, a_n)\|_1^*$ for all $(a_1, \ldots, a_n) \in \mathbb{R}^n \cong V^*$.

4.(a) For every integer $n \geq 1$ and every pair of values p, q with $1 \leq p \leq q \leq \infty$, find the largest contsant $c \in \mathbb{R}$ and the smallest constant $C \in \mathbb{R}$ such that $c\|x\|_p \leq \|x\|_q \leq C\|x\|_p$ for all $x \in \mathbb{C}^n$.

(b) If A is an $n \times n$ complex matrix, we interpret it as a linear map $\mathbb{C}^n \to \mathbb{C}^n$ in the usual way. We use the usual Euclidean norm $\|\cdot\|_2$ on $\mathbb{C}^n$. For every integer $n \geq 1$, find the largest contsant $c \in \mathbb{R}$ and the smallest constant $C \in \mathbb{R}$ such that

$$c\|A\|_{\text{op}} \leq \max_{1 \leq i,j \leq n} |A_{ij}| \leq C\|A\|_{\text{op}}$$

holds for all matrices A.

[Specifically, $\|A\|_{\text{op}}$ is the smallest constant K such that $\|Ax\|_2 \leq K\|x\|_2$ for all $x \in \mathbb{C}^n$.]

5. Let V be a finite-dimensional vector space.

(a) Suppose $\|\cdot\|$, $\|\cdot\|'$ are two norms on V such that $\|\cdot\| \leq \|\cdot\|'$. Prove that $\|\cdot\|'^* \leq \|\cdot\|^*$.

(b) If $\|\cdot\|$ is a norm on V and $\alpha \in \mathbb{R}_{>0}$ is a scalar, write $\alpha\|\cdot\|$ for the norm $v \mapsto \alpha\|v\|$. Prove that the dual norm to $\alpha\|\cdot\|$ is $(1/\alpha)\|\cdot\|^*$.