MATH 202A
APPLIED ALGEBRA I
FALL 2021

Homework week 8

Due by 23:59 on Sunday 21st November (hand in via Gradescope).

For two norms $\|\cdot\|$, $\|\cdot\|'$ on the same vector space V, we write $\|\cdot\| \le \|\cdot\|'$ to mean $\|v\| \le \|v\|'$ for all $v \in V$.

Throughout, you may use any form on the Hahn–Banach theorem that has been stated in lectures (or a form of the Hahn–Banach theorem that has not been stated in lectures, if you want, provided it does not trivialize the problem).

1. Let V, W be finite-dimensional normed spaces, with the norms denoted $\|\cdot\|_V$, $\|\cdot\|_W$. Consider the dual vector spaces V^*, W^* as normed spaces with the norms $\|\cdot\|_V^*$, $\|\cdot\|_W^*$ respectively.

Also, suppose $\phi\colon V \to W$ is a linear map and $\phi^*\colon W^* \to V^*$ is the dual map.

(a) Prove the operator norm bound $\|\phi^*\|_{W^* \to V^*} \le \|\phi\|_{V \to W}$.

(b) Prove that $\|\phi^*\|_{W^* \to V^*} = \|\phi\|_{V \to W}$.

[**Hint**: stating part (a) before part (b) is a hint.]

(c) Reflect on the relationship between your proof of Pset 4 Q3 on the one hand, and (a) here together with Example 5.3.1 on the other hand. [You do not need to write anything for this part.]

2. Let V be a finite-dimensional vector space and $\|\cdot\|$, $\|\cdot\|'$ two norms on V. Consider the following functions $V \to \mathbb{R}$:—

$$v \mapsto \|v\| + \|v\|'$$
$$v \mapsto \max(\|v\|, \|v\|')$$
$$v \mapsto \inf_{\substack{x,y \in V \\ x+y=v}} \left(\|x\| + \|y\|'\right)$$
$$v \mapsto \inf_{\substack{x,y \in V \\ x+y=v}} \max(\|x\|, \|y\|').$$

We denote there respectively by $\|\cdot\|+\|\cdot\|'$, $\max(\|\cdot\|, \|\cdot\|')$, $\operatorname{coplus}(\|\cdot\|, \|\cdot\|')$ and $\operatorname{comax}(\|\cdot\|, \|\cdot\|')$. Note these last two are non-standard notation.

(a) Very briefly, verify that these are all norms on V.

(b) Prove (directly, using the definitions) that

$$\left(\|\cdot\| + \|\cdot\|'\right)^* \le \operatorname{comax}(\|\cdot\|^*, \|\cdot\|'^*)$$

and

$$\left(\max(\|\cdot\|, \|\cdot\|')\right)^* \le \operatorname{coplus}(\|\cdot\|^*, \|\cdot\|'^*).$$

(c) Prove (directly, using the definitions) that $\operatorname{coplus}(\|\cdot\|, \|\cdot\|') \le \|\cdot\|$ and $\operatorname{coplus}(\|\cdot\|, \|\cdot\|') \le \|\cdot\|'$.

(d) Prove that $\left(\operatorname{comax}(\|\cdot\|, \|\cdot\|')\right)^* \ge \|\cdot\|^* + \|\cdot\|'^*$.

[**Hint:** Given $\phi \in V^*$, by definition you are trying to find $v \in V$ such that $|\phi(v)|$ is large but $\|v\|_{\text{comax}}$ is small. Note this is the same as trying to find $x, y \in V$ with $|\phi(x+y)|$ large but $\max(\|x\|, \|y\|')$ small. You can do this directly from the definitions of $\|\phi\|^*$ and $\|\phi\|'^*$.]

(e) Prove that in fact equality holds in (b).

[**Hint:** now you shouldn't have to do more work.]

3. Consider the vector space $\mathbb{C}^n$ with the usual ℓ^p-norms.

(a) Prove that $\| \cdot \|_2 \leq \frac{1}{2} \| \cdot \|_{4/3} + \frac{1}{2} \| \cdot \|_4$.

[Here we use the notation developed in previous questions.]

(b) Let $x \in \mathbb{C}^n$ be a vector with $\|x\|_2 \leq 1$. Prove that there is a decomposition $x = y + z$ where $y, z \in \mathbb{C}^n$ satisfy $\|y\|_{4/3} \leq 1/2$ and $\|z\|_4 \leq 1/2$.

[**Note**: you could give a direct proof of this statement, i.e. by finding an explicit formula for y_i and z_i, using calculus or otherwise. This is strongly discouraged. You should attempt to obtain this fact "for free" from (a), by leveraging the results from the rest of the pset.]

4. Let $V = \mathbb{R}^n$ with the usual inner product and $\| \cdot \|_p$-norms, and let $U \subseteq V$ be some subspace. For $v \in V$, define

$$f(v) = \inf_{u \in U} \|u + v\|_1$$

and

$$g(v) = \sup_{\substack{w \in U^\perp \\ \|w\|_\infty \leq 1}} \langle v, w \rangle.$$

(a) Prove directly that $f(v) \geq g(v)$ for all $v \in V$.

(b) Prove that $f|_{U^\perp}$ and $g|_{U^\perp}$ are norms on $U^\perp$.

(c) Prove the following statement: for all $w \in U^\perp$ there exists $\phi \in U^\perp$ such that $|\langle w, \phi \rangle| \geq \|w\|_\infty f(\phi)$.

[**Hint:** take the fact that $\| \cdot \|_1$, $\| \cdot \|_\infty$ are dual norms, and apply it to w to construct a suitable ϕ.]

(d) Hence, or otherwise, prove that $f = g$.

[**Note:** in this case, it is useful to reflect on how hard it is to prove (d) without the Hahn–Banach theorem.]