Pebbles
and
Expansions in the Polynomial Ring

by

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Astract We show here that some simple combinatorial facts concerning arrangements of pebbles on an $n \times n$ board have surprising consequences in the study of expansions in the polynomial ring $\mathbf{Q}[x_1, x_2, \ldots, x_n]$. In particular in this manner we obtain a purely combinatorial proof of an identity of Lascoux-Schutzenberger given in Funkt. Anal **21** (1987) 77-78.

I. Introduction

We recall that the "*lattice cells*" of the Cartesian plane are the subsets

$$C_{i,j} \;=\; \{(x,y) : i-1 \leq x \leq i \;\;\&\;\; j-1 \leq x \leq j\,\}$$

The "*$n \times n$ board*" is the collection of lattice cells

$$\mathcal{B}_n \;=\; \{C_{i,j} \,:\, 1 \leq i, j \leq n\,\}\,.$$

The "*diagonal* " of the board is the collection

$$\mathcal{D}_n \;=\; \{C_{i,i} \,:\, 1 \leq i \leq n\,\}$$

We will deal here with arrangements of pebbles in the $n \times n$ board with the following properties:

(1) *At most one pebble in each cell and no pebbles in the diagonal.*

(2) *The pebbles above the diagonal are white and arranged in left justified rows, some of which may be empty.*

(2) *Below the diagonal the pebbles are black but unrestricted.*

The display below illustrates such an arrangement in $\mathcal{B}_8$. For clarity we have shaded the diagonal cells. The numbers on the left of the board indicate the row counts of the white pebbles and the numbers below the board indicate the column counts of the black pebbles.

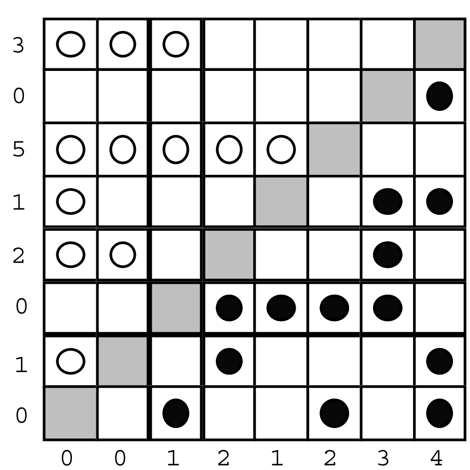


Fig. I.1

Given such a filling $\mathcal{F}$ the vectors which give the row counts of white pebbles and column counts of the black cells will be respectively denoted $\alpha(\mathcal{F})$ and $\beta(\mathcal{F})$. For the filling in Fig. 1 we have

$$\alpha(\mathcal{F}) = (0,1,0,2,1,5,0,3) \quad \text{and} \qquad \beta(\mathcal{F}) = (0,0,1,2,1,2,3,4)$$

It will be convenient to set for any vector of integers $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$

$$|\alpha| \;\; = \;\; \alpha_1 + \alpha_2 + \cdots + \alpha_n \,.$$

We shall say that a vector of integers $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_n)$ is "*subtriangular*" if

$$0 \leq \alpha_i \leq i-1 \qquad\qquad (\text{ for } i = 1, \ldots, n).$$

Clearly for all our board fillings $\mathcal{F}$ both $\alpha(\mathcal{F})$ and $\beta(\mathcal{F})$ are subtriangular vectors in particular we necessarily have also the inequalities

$$|\alpha(\mathcal{F})| \;\leq\; \binom{n}{2}\,, \qquad\qquad |\beta(\mathcal{F})| \;\leq\; \binom{n}{2}\,. \tag{I.1}$$

Given two subtriangular vectors $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ and $\beta = (\beta_1, \beta_2, \ldots, \beta_n)$ we shall denote by $\mathcal{C}(\alpha, \beta)$ the collection of all fillings of $\mathcal{B}_n$ satisfying conditions (1), (2) and (3) and whose white row counts and black row counts are given by α and β respectively. In symbols

$$\mathcal{C}(\alpha,\beta) \;\; = \;\; \big\{\, \mathcal{F} \,:\, \alpha(\mathcal{F}) = \alpha \;\; \& \;\; \beta(\mathcal{F}) = \beta \,\big\} \tag{I.2}$$

We should note that the cardinality of the collection $\mathcal{C}(\alpha, \beta)$ is given by a product of binomial coefficients. More precisely we have

$$\#\mathcal{C}(\alpha,\beta) \;\; = \;\; \prod_{i=1}^{n} \binom{i-1}{\beta_i} \tag{I.3}$$

Given a filling $\mathcal{F}$ we let $\tau_i(\mathcal{F})$ denote the total number of pebbles (white or black) in the i^{th} row of the board. We set

$$\tau(\mathcal{F}) = (\tau_1(\mathcal{F}), \tau_2(\mathcal{F}), \ldots, \tau_n(\mathcal{F}))$$

and call it "*the vector of total row counts*". For the example in Fig. 1 we have

$$\tau(\mathcal{F}) \;\; = \;\; (3,1,5,3,3,4,3,3) \tag{I.4}$$

It will be convenient to set

$$\delta \;\; = \;\; (0,1,2,3,\ldots,n-1)\,. \tag{I.5}$$

This given we shall say that two subtriangular vectors $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ and $\beta = (\beta_1, \beta_2, \ldots, \beta_n)$ are "*complementary*" is and only if

$$\beta \;\; = \;\; \delta - \alpha \tag{I.6}$$

that is

$$\beta_i \;=\; i-1-\alpha_i \qquad \text{for } i=1,2,\ldots,n$$

The contents of these notes are divided into three sections. In the first section we establish some properties of the vectors of total counts for the elements of our collections $\mathcal{C}(\alpha,\beta)$. In the second section we derive consequences of these properties for polynomials in $\mathbf{Q}[x_1,x_2,\ldots,x_n]$ and in the third section we give some applications.

Acknowledgement

We are grateful to Jeff Remmel for several stimulating exchanges and invaluable help in the proof of Theorem 1.1.

1. Pebbles' magics.

To simplify our presentation it will be good to make some preliminary remarks and introduce further terminology. To begin, given a filling $\mathcal{F}$ of the board $\mathcal{B}_n$ we shall denote by $\mathcal{F}_{\leq i}$ the filling of the board $\mathcal{B}_i$ obtained by removing the last $n-i$ rows and columns of $\mathcal{F}$. Note that since there are no pebbles on the diagonal, there can't more than $n-1$ total pebbles falling in any row of our fillings. Thus if a $\tau(\mathcal{F})$ has distinct components these can only be the integers $0,1,2,3,\ldots,n-1$ in some order. We shall then say that $\tau(\mathcal{F})$ is a "*permutation of* δ " or briefly a "*permutation*" and $\mathcal{F}$ itself will be called "*permutational*". A filling $\mathcal{F}$ of $\mathcal{B}_n$ will be called "*perfect*" if the fillings

$$\mathcal{F}_{\leq 2}\ ,\ \mathcal{F}_{\leq 3}\ ,\ \ldots\ ,\ \mathcal{F}_{\leq n}$$

are all permutational. If $\mathcal{F}$ is permutational and $\sigma=(\sigma_1,\sigma_2,\ldots,\sigma_n)$ is the permutation giving

$$\tau(\mathcal{F})=(\delta_{\sigma_1},\delta_{\sigma_2},\ldots,\delta_{\sigma_n})$$

then the sign of σ will be briefly referred as the "*sign of* $\mathcal{F}$". In symbols

$$sign(\mathcal{F})=sign(\sigma) \tag{1.1}$$

This given, the total pebble counts have the following remarkable properties:

Theorem 1.1

Let $\alpha=(\alpha_1,\alpha_2,\ldots,\alpha_n)$ and $\beta=(\beta_1,\beta_2,\ldots,\beta_n)$ be any given subtriangular vectors, then

(a) *Every $\tau(\mathcal{F})$ with distinct components must be a permutation of δ.*

(b) *If $|\alpha|+|\beta|\neq|\delta|$, the total count vectors $\tau(\mathcal{F})$ have repeated components for all $\mathcal{F}\in\mathcal{C}(\alpha,\beta)$.*

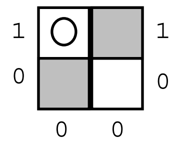
(c) *If $\beta=\delta-\alpha$ then the collection $\mathcal{C}(\alpha,\beta)$ has a unique permutational filling $\mathcal{F}$. This filling is also perfect and we have*

$$sign(\mathcal{F})=(-1)^{\beta_1+\beta_2+\ldots+\beta_n}\ ,. \tag{1.2}$$

(d) *If $|\alpha|+|\beta|=|\delta|$ but $\beta\neq\delta-\alpha$ there may be more than one permutational filling in $\mathcal{C}(\alpha,\beta)$. However in this case the signs of the fillings always add up to zero.*

Proof

We have observed above that Property *(a)* holds true. Property *(b)* is an immediate consequence of *(a)*. To prove c) we start by looking at the case $n=2$. Here there are only two complementary pairs, namely $\big((0,1),(0,0)\big)$ and $\big((0,0),(0,1)\big)$. The two collections $\mathcal{C}\big((0,1),(0,0)\big)$ and $\mathcal{C}\big((0,0),(0,1)\big)$ reduce to the single fillings given in the figure below.



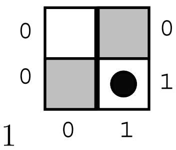


Fig 1.1

For the first filling we have $\tau(\mathcal{F})=(0,1)=\delta$ and for the second $\tau(\mathcal{F})=(1,0)=(1,2)\,\delta$. So the sign, in each case, is as stated in 1.2 and thus property *(c)* holds true in this case. We can thus proceed by induction. We shall begin by showing that when $\beta=\delta-\alpha$ we can easily construct a filling $\mathcal{F}_o\in\mathcal{C}(\alpha,\beta)$ that is not only permutational but also perfect. We then show that $\mathcal{F}_o$ is the only permutational filling in $\mathcal{C}(\alpha,\beta)$. Since both fillings in Fig. 1.1 are trivially perfect let us assume that for any choice of complementary $\alpha'=(\alpha_1,\alpha_2,\ldots\alpha_{n-1})$ and $\beta'=(\beta_1,\beta_2,\ldots,\beta_{n-1})$ we have constructed a perfect filling $\mathcal{F}'_o\in\mathcal{C}(\alpha',\beta')$. Let α_n,β_n be given with $\alpha_n+\beta_n=n-1$ and let $\alpha=(\alpha_1,\alpha_2,\ldots\alpha_{n-1},\alpha_n)$, $\beta=(\beta_1,\beta_2,\ldots,\beta_{n-1},\beta_n)$ For convenience set $\alpha_n=k$ and $\beta_n=n-1-k$. Note that to complete the induction we need only construct a permutational filling $\mathcal{F}_o\in\mathcal{C}(\alpha,\beta)$ such that

$$\mathcal{F}_{\le n-1}=\mathcal{F}'_o\,. \tag{1.3}$$

These conditions uniquely determine $\mathcal{F}_o$. In fact, to satisfy 1.3 , we must construct $\mathcal{F}_o$ by filling the subboard $\mathcal{B}_{n-1}$ of $\mathcal{B}_n$ with $\mathcal{F}_{\le n-1}$ then place k left justified white pebbles in the top row of $\mathcal{B}_n$ and $n-1-k$ black pebbles in the last column of $\mathcal{B}_n$. Now the requirement that $\tau(\mathcal{F}_o)$ be a permutation of $(0,1,\ldots,n-1)$ determines the rows where these black pebbles must be placed. The reason for this is simple. Given that $\alpha_n=k$, we necessarily have $\tau_n(\mathcal{F})=k$ for all $\mathcal{F}\in\mathcal{C}(\alpha,\beta)$. So we need only assure that

$$\tau_1(\mathcal{F}_o),\tau_2(\mathcal{F}_o),\ldots,\tau_{n-1}(\mathcal{F}_o)$$

are a permutation of

$$0,1,\ldots,k-1,k+1,\ldots,n-1\,.$$

Since 1.3 yields that $\tau_i(\mathcal{F}_o)=\tau_i(\mathcal{F}'_o)+{0\atop 1}$ we see that we are forced to place the black pebbles in the rows where the components of $\tau_i(\mathcal{F}'_o)$ take the values

$$k,k+1,\ldots,n-2\,.$$

This completes our induction and the proof that perfect fillings always exist and are unique. However the possibility remains that there may be permutational fillings that are not perfect.

To exclude this possibility we need to familiarize ourselves with the nature of our perfect fillings. This is best done with an example that illustrates the algorithm which transpires from our existence proof. In fact, for a given $\alpha=(\alpha_1,\alpha_2,\ldots,\alpha_n)$ our inductive process determines the sequence of steps we must follow to construct the successive fillings

$$\mathcal{F}_{\leq 2}(\mathcal{F}_o)\rightarrow\mathcal{F}_{\leq 3}(\mathcal{F}_o)\rightarrow\mathcal{F}_{\leq 4}(\mathcal{F}_o)\rightarrow\cdots\rightarrow\mathcal{F}_{\leq n-1}(\mathcal{F}_o)\rightarrow\mathcal{F}(\mathcal{F}_o)\,.$$

The display below illustrates the sequence we obtain for $\alpha=(0,1,0,3,1,2)$.

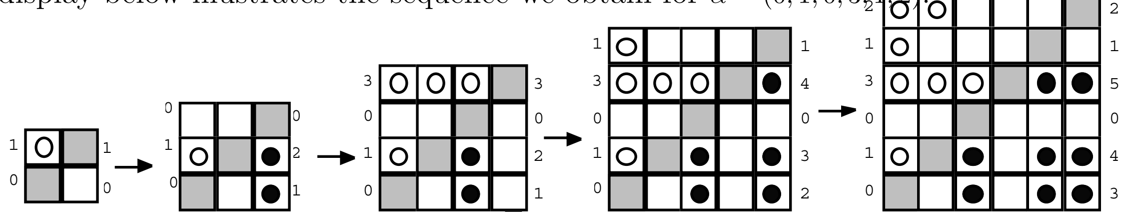


Fig. 1.2

The information contained in such a display can be compressed into a triangular array given by the sequence of vectors

$$\tau(\mathcal{F}_{\leq 2})\,,\tau(\mathcal{F}_{\leq 3})\,,\ldots\tau(\mathcal{F}_{\leq n-1})\,,\tau(\mathcal{F})\,, \tag{1.4}$$

placed from bottom to top under the diagonal of $\mathcal{B}_n$. In this manner the sequence of fillings displayed in Fig 1.2 reduces to the single diagram given below

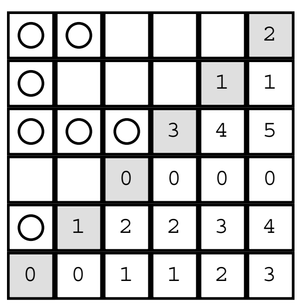


Fig. 1.3

Clearly we can recover the successive fillings in Fig. 1.2 by placing a pebble in a cell whenever the integer in the cell is greater than the integer in the cell immediately to the left. We should also note that our argument shows that the triangular arrays resulting from a perfect filling may be directly constructed from a very simple recursion. In fact, if $c(i,j)$ (for $i\leq j$) denotes the content of cell (i,j), then the triangular array corresponding to a given subtriangular vector $\alpha=(\alpha_1,\alpha_2,\ldots,\alpha_n)$ is obtained by setting

$$c(i,i)=\alpha_i \qquad \text{and for } j>i \;\; c(i,j)=c(i,j-1)+\begin{cases}0 & \text{if } c(i,j-1)<\alpha_j\,,\\ 1 & \text{if } c(i,j-1)\geq\alpha_j\,.\end{cases} \tag{1.5}$$

In particular we see that the contents of row i only depend on the components

$$\alpha_i, \alpha_{i+1}, \ldots, \alpha_n \,.$$

Keeping this in mind we shall complete our proof of Property *(c)* by induction on n. Let us assume then that for any two complementary subtriangular vectors $\alpha' = (\alpha'_1, \alpha'_2, \ldots, \alpha'_{n-1})$ and $\beta' = (\beta'_1, \beta'_2, \ldots, \beta'_{n-1})$ there is one and only one permutational filling $\mathcal{F}'_o$ in $\mathcal{C}(\alpha', \beta')$ and its sign is as given in 1. By what we have shown it follows that $\mathcal{F}'_o$ must be perfect. Now let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ and let $\beta = \delta - \alpha$, and suppose that the filling $\mathcal{F} \in \mathcal{C}(\alpha, \beta)$ is permutational. Note that the position of the component $n-1$ in $\tau(\mathcal{F})$ is uniquely determined by α. To see this note that either we have $\alpha_i < i-1$ for all i or for some $k \geq 2$ we have

$$\alpha_k = k-1 \quad \text{and} \qquad \alpha_j < j-1 \quad \forall \quad j = k+1, k+2, \ldots, n$$

Now in the first case we must have $\tau_n(\mathcal{F}) = n-1$ and in the second case $\tau_k(\mathcal{F}) = n-1$. In fact, in the first case the total pebble counts on rows $2, 3, \ldots, n$ cannot exceed $n-2$. The same must hold true in the second case for the same reason for rows $k+1, k+2, \ldots, n$. On the other hand, also for rows $1, 2, \ldots, k-1$ the counts cannot exceed $n-2$ because $\alpha_k = k-1$ forces $\beta_k = 0$ which in turn prevents placing any black pebbles in column k. Thus in any case the position of $n-1$ is uniquely determined by α as asserted.

Proceeding in the second case we construct a filling $\mathcal{F}'$ of $\mathcal{B}_{n-1}$ by removing row k entirely from $\mathcal{F}$ together with the $k-1$ empty cells below the diagonal in column k and $n-k$ empty cells immediately above the diagonal in columns

$$k+1, k+2, \ldots, n \,.$$

It is easily seen that the resulting filling $\mathcal{F}'$ is in $\mathcal{C}(\alpha', \beta')$ with

$$\alpha' = (\alpha_1, \alpha_2, \ldots, \alpha_{k-1}, \alpha_{k+1}, \ldots, \alpha_n) \quad \text{and} \qquad \beta' = (\beta_1, \beta_2, \ldots, \beta_{k-1}, \beta_{k+1} - 1, \ldots, \beta_n - 1)$$

moreover α' and β' are complementary and we must also have

$$\tau_1(\mathcal{F}') = \tau_1(\mathcal{F}), \tau_2(\mathcal{F}') = \tau_2(\mathcal{F}) \;,\; \ldots, \tau_{k-1}(\mathcal{F}') = \tau_{k-1}(\mathcal{F}), \tau_k(\mathcal{F}') = \tau_{k+1}(\mathcal{F}), \ldots, \tau_{n-1}(\mathcal{F}') = \tau_n(\mathcal{F}) \qquad 1.6$$

In other words $\tau(\mathcal{F}')$ is simply obtained by removing the component $n-1$ from $\tau(\mathcal{F})$. It thus follows that $\mathcal{F}'$ is a permutational filling of $\mathcal{C}(\alpha', \beta')$. By induction $\mathcal{F}'$ must also be perfect. It then follows from the recursions in 1.5 that the top $n-k$ rows of the triangular arrays for α' and α must be identical. This gives that

$$\tau_k(\mathcal{F}') = \tau_{k+1}(\mathcal{F}_o) \;, \qquad \tau_{k+1}(\mathcal{F}') = \tau_{k+2}(\mathcal{F}_o) \;,\; \ldots \;,\; \tau_{n-1}(\mathcal{F}') = \tau_n(\mathcal{F}_o) \qquad 1.7$$

with $\mathcal{F}_o$ the perfect filling of $\mathcal{C}(\alpha, \beta)$. From the recursions in 1.5 it also follows that the first $k-1$ entries in rows $1, 2, \ldots, k-1$ of the triangular arrays for α' and α must also be identical. Since there can't be any black pebbles in column k of $\mathcal{F}_o$, it follows that the partial row sums for $\mathcal{F}_o$ can't change across column k. But after that the recursions will force them to change precisely as the partial row sums change for α'. This shows that we must also have

$$\tau_1(\mathcal{F}') = \tau_1(\mathcal{F}_o)\ , \quad \tau_2(\mathcal{F}') = \tau_2(\mathcal{F}_o)\ ,\ \ldots\ ,\ \tau_{k-1}(\mathcal{F}') = \tau_{k-1}(\mathcal{F}_o) \tag{1.8}$$

Since both $\tau_k(\mathcal{F}_o)$ and $\tau_k(\mathcal{F})$ must necessarily be equal to $n-1$, we see that combining 1.7 and 1.8 with 1.6 we derive that

$$\mathcal{F} \;=\; \mathcal{F}_o$$

This proves that $\mathcal{F}$ is perfect. To complete the induction in this case we need only check that the sign is correct. To this end note that by induction we must have

$$sign(\mathcal{F}') = (-1)^{\beta_1+\beta_2+\cdots+\beta_{k-1}+\beta_{k+1}+\cdots+\beta_n-(n-k)}\,. \tag{1.9}$$

Since $\tau_k(\mathcal{F}) = n-1$, it takes exactly $n-k$ adjacent transpositions to move $n-1$ to the end of $\tau(\mathcal{F})$, after doing that we obtain a vector that is $\tau(\mathcal{F}')$ concatenated with $n-1$. Thus

$$sign(\mathcal{F}) = (-1)^{n-k} sign(\mathcal{F}') \tag{1.10}$$

since $\beta_k = 0$, 1.9 and 1.10 give 1.2. This completes the proof of *(c)* in this case. The other case can essentially be dealt in the same manner by specialiizing $k=1$ in the various steps of the argument.

We are left to prove Property *(d)*. To begin let us suppose that we have a permutational filling $\mathcal{F} \in \mathcal{C}(\alpha,\beta)$ and suppose further that $\beta \neq \delta - \alpha$. This given we claim that for at least one $i = 2, 3, \ldots, n-1$ the vector $\tau(\mathcal{F}_{\leq i})$ must have repeated components. To see this note that since all the components of $\tau(\mathcal{F}_{\leq i})$ are $\leq i-1$, if $\tau(\mathcal{F}_{\leq i})$ has distinct components it must then be a rearrangement of $(0, 1, \ldots, i-1)$. But that cannot happen for all i because then we have

$$|\tau(\mathcal{F}_{\leq i})| - |\tau(\mathcal{F}_{\leq i-1})| \;=\; i-1\,.$$

Now this difference is also given by the number of white pebbles in row i of $\mathcal{F}_{\leq i}$ plus the number of black pebbles in column i of $\mathcal{F}_{\leq i}$. But the former is α_i and the latter is β_i. This would yield

$$\alpha_i + \beta_i = i-1 \qquad \text{for all } i = 2, 3, \ldots, n$$

which is excluded by our assumption that $\beta \neq \delta - \alpha$.

Suppose then that i_o is the largest i for which $\tau(\mathcal{F}_{\leq i})$ has repeated components. And suppose as well that h, k are the largest pair for which $\tau_h(\mathcal{F}_{\leq i_o}) = \tau_k(\mathcal{F}_{\leq i_o})$ and let $\phi(\mathcal{F})$ be the new filling obtained by interchanging the tail ends of rows h and k of $\mathcal{F}$ that are to the right of column i_o. This construction defines a map $\mathcal{F} \rightarrow \phi(\mathcal{F})$ with the following three properties

(1) $\mathcal{F} \in \mathcal{C}(\alpha,\beta) \;\Rightarrow\; \phi(\mathcal{F}) \in \mathcal{C}(\alpha,\beta)\,.$
(2) ϕ *is an involution of* $\mathcal{C}(\alpha,\beta)$, *that is* $\phi(\phi(\mathcal{F})) = \mathcal{F} \quad \forall \mathcal{F} \in \mathcal{C}(\alpha,\beta)$.
(3) $sign\big(\phi(\mathcal{F})\big) = -sign(\mathcal{F})$

where for convenience we make the convention that the sign of a vector with repeated components is assigned the value zero.

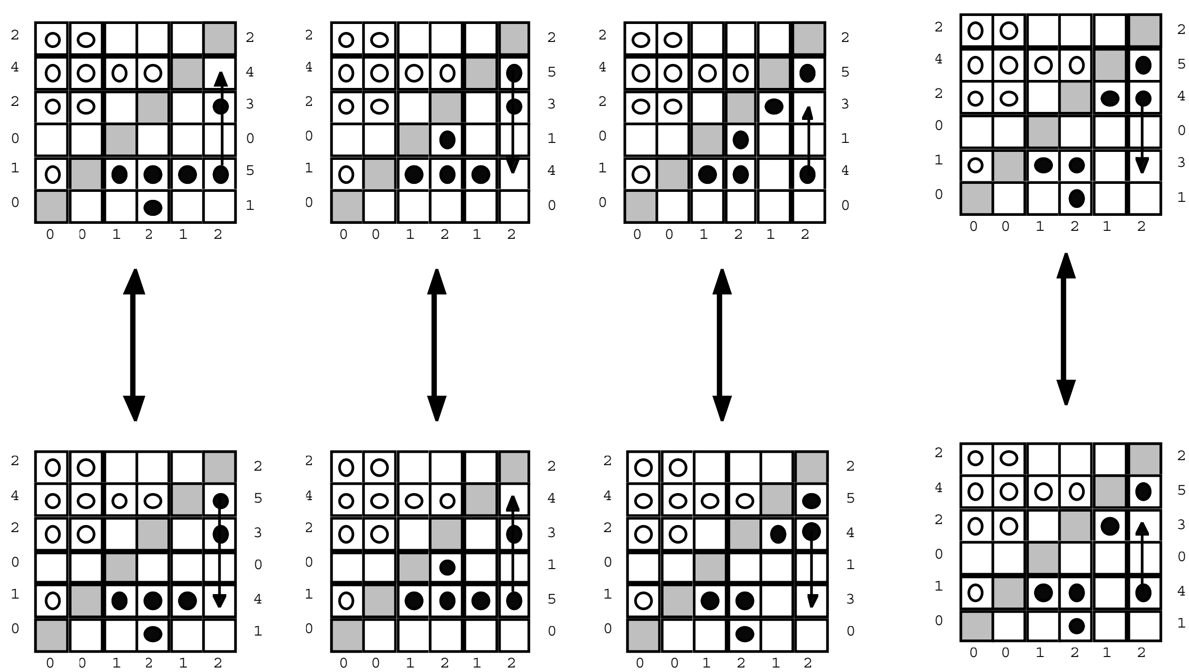
Property *(1)* is clear since the interchange of row tails preserves the column counts of black pebbles.

Property *(2)* follows from the fact that the interchange of row tails preserves the $i's$ for which $\tau_h(\mathcal{F}_{\leq i})$ has or has not distinct components, as well as the largest pair h, k for which the equality $\tau_h(\mathcal{F}_{\leq i_o}) = \tau_k(\mathcal{F}_{\leq i_o})$ holds true. .

Finally property *(3)* is assured by the fact that $\tau_h(\phi(\mathcal{F})) = \tau_k(\mathcal{F})$ and $\tau_k(\phi(\mathcal{F})) = \tau_h(\mathcal{F})$. So when $\tau(\mathcal{F})$ is a rearrangement of δ the map ϕ changes the sign of the corresponding permutation.

In summary this involution produces a pairing of the elements of $\mathcal{C}(\alpha, \beta)$ which couples an element $\mathcal{F}$ for which $\tau(\mathcal{F})$ has positive sign with one with negative sign, forcing the signs to add up to zero as asserted. This completes the proof of the Theorem.

In the case $\alpha = (0, 1, 0, 2, 4, 2)$ and $\beta = (0, 0, 1, 2, 1, 2)$ we found only 8 fillings whose total row counts are rearrangements of $(0, 1, 2, 3, 4, 5)$. These are displayed below along with the pairing induced by ϕ.



2. A remarkable scalar product for $\mathbf{Q}[\mathbf{X_n}]$

In the space of polynomials in $x_1, x_2, \ldots, x_n$ with rational coefficients we introduce a scalar product "$\langle\ ,\ \rangle$" by setting for $x^p = x_1^{p_1} x_2^{p_2} \cdots x_n^{p_n}$ and $x^q = x_1^{q_1} x_2^{q_2} \cdots x_n^{q_n}$

$$\big\langle x^p\ ,\ x^q \big\rangle \;=\; \frac{det\, \big\| x_i^{p_j+q_j} \big\|_{i,j=1}^n}{det\, \big\| x_i^{j-1} \big\|_{i,j=1}^n} \qquad 2.1$$

Given a symmetric polynomial P it will be convenient to use plethystic notation and denote $P[x_1, x_2, \ldots, x_k]$ by the symbol "$P[X_k]$" with $X_k = x_1 + x_2 + \cdots + x_k$.

This section is dedicated to the proof of the following result

Theorem 2.1

For a subtriangular vector $\beta = (\beta_1, \beta_2, \ldots, \beta_n)$ set

$$\phi_\beta[x_1, x_2, \ldots, x_n] \;=\; (-1)^{|\beta|} e_{b_2}[X_1] e_{\beta_3}[X_2] e_{\beta_4}[X_3] \cdots e_{\beta_n}[X_{n-1}]\,. \qquad 2.2$$

This given, for any subtriangular vector $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ we have

$$\big\langle x^\alpha\ ,\ \phi_\beta \big\rangle \;=\; \begin{cases} 1 & \textit{if } \beta = \delta - \alpha\ , \\ 0 & \textit{if } \beta \neq \delta - \alpha\ . \end{cases} \qquad 2.3$$

Proof

Note that for $\alpha = (0, 1, 0, 2, 1, 5, 0, 3)$ and $\beta = (0, 0, 1, 2, 1, 2, 3, 4)$ we have

$$x^\alpha = x_2 x_4^2 x_5 x_6^5 x_8^3$$

and

$$\phi_\beta = -e_1[X_2] e_2[X_3] e_1[X_4] e_2[X_5] e_3[X_6] e_4[X_7]\,.$$

Now a typical monomial occurring in the expansion of ϕ_β is of the form

$$x_a \times (x_{b_1} x_{b_2}) \times (x_c) \times (x_{d_1} x_{d_2}) \times (x_{e_1} x_{e_2} x_{e_3}) \times (x_{f_1} x_{f_2} x_{f_3} x_{f_4}) \qquad 2.4$$

with

$$1 \leq a \leq 2\,,\;\; 1 \leq b_1 < b_2 \leq 3\,,\;\; 1 \leq c \leq 4\,,\;\; 1 \leq d_1 < d_2 \leq 5\,,\;\; 1 \leq e_1 < e_2 < e_3\,, 6\,,\;\;\; 1 \leq f_1 < f_2 < f_3 < f_4 \leq 7\,,$$

In particular one of these monomials is

$$x^q = x_1^3 x_2^2 x_3^4 x_4 x_5^2 x_7 \;=\; x_1 \times (x_2 x_3) \times (x_3) \times (x_1 x_3) \times (x_3 x_4 x_5) \times (x_1 x_2 x_5 x_7)\,. \qquad 2.5$$

A moment of reflection should reveal that we can represent each of the monomials in 2.4 as a placement of black pebbles in $\mathcal{B}_8$. That is the factor x_a is represented by a pebble in column 3 and row a, the factor $x_{b_1} x_{b_2}$ by two pebbles in column 4 in rows b_1 and b_2, etc. , and finally the factor $x_{f_1}x_{f_2}x_{f_3}x_{f_4}$ by 4 pebbles in column 8 and rows f_1, f_2, f_3, f_4. At the same time the monomial x^α can be represented by a placement of $\alpha_2, \alpha_3, \ldots, \alpha_n$ left justified white pebbles in rows $2, 3, \ldots, n$. In particular, we see in this manner that the pair

$$x^\alpha = x_2 x_4^2 x_5 x_6^5 x_8^3 \, , \qquad x^q = x_1^3 x_2^2 x_3^4 x_4 x_5^2 x_7$$

may be represented by the filling displayed in Fig. 1. Now the scalar product

$$\langle\, x^\alpha \, , \, \phi_\beta \rangle = \langle x_2 x_4^2 x_5 x_6^5 x_8^3 \, , \, e_1[X_2] e_2[X_3] e_1[X_4] e_2[X_5] e_3[X_6] e_4[X_7] \,\rangle$$

is a sum of terms of the form

$$\big\langle x_2 x_4^2 x_5 x_6^5 x_8^3 \, , \, x_a \times (x_{b_1} x_{b_2}) \times (x_c) \times (x_{d_1} x_{d_2}) \times (x_{e_1} x_{e_2} x_{e_3}) \times (x_{f_1} x_{f_2} x_{f_3} x_{f_4}) \big\rangle \, .$$

In particular one of the terms will be

$$\big\langle x^\alpha \, , \, x^q \big\rangle \;=\; \frac{det\| x_i^{\alpha_j + q_j} \|_{i,j=1}^8}{det\| x_i^{j-1} \|_{i,j=1}^8} \;=\; \frac{det \begin{pmatrix} x_1^3 & x_1^3 & x_1^4 & x_1^3 & x_1^3 & x_1^5 & x_1 & x_1^3 \\ x_2^3 & x_2^3 & x_3^4 & x_2^3 & x_2^3 & x_2^5 & x_2 & x_3^2 \\ x_3^3 & x_3^3 & x_3^4 & x_3^3 & x_3^3 & x_3^5 & x_3 & x_3^3 \\ x_4^3 & x_4^3 & x_4^4 & x_4^3 & x_4^3 & x_4^5 & x_4 & x_4^3 \\ x_5^3 & x_5^3 & x_5^4 & x_5^3 & x_5^3 & x_5^5 & x_5 & x_5^3 \\ x_6^3 & x_6^3 & x_6^4 & x_6^3 & x_6^3 & x_6^5 & x_6 & x_6^3 \\ x_7^3 & x_7^3 & x_7^4 & x_7^3 & x_7^3 & x_7^5 & x_7 & x_7^3 \\ x_8^3 & x_8^3 & x_8^4 & x_8^3 & x_8^3 & x_8^5 & x_8 & x_8^3 \end{pmatrix}}{det\| x_i^{j-1} \|_{i,j=1}^8} \;=\; 0 \, .$$

We should be able to see now how our study of arrangements of pebbles helps in the evaluation of the scalar product in 2.3. In fact, in full generality, each of the terms obtained in the expansion of the scalar product $\langle x^\alpha \, , \, \phi_\beta \rangle$ will correspond to a placement of pebbles in $\mathcal{B}_n$. A term corresponding to an $\mathcal{F} \in \mathcal{C}(\alpha, \beta)$ where $\tau(\mathcal{F})$ has repeated components will yield a determinant with two equal columns and thus contributes nothing to the sum. So the only surviving terms are those represented by an $\mathcal{F} \in \mathcal{C}(\alpha, \beta)$ where $\tau(\mathcal{F})$ has distinct components. But, Theorem 1.1 asserts that this can only happen when $|\alpha| + |\beta| = \binom{n}{2}$, and $\tau(\mathcal{F})$ is a rearrangement of δ. But then the determinant in the numerator of $\langle x^\alpha \, , \, x^{\tau(\mathcal{F})} \rangle$ will evaluate to a multiple of the denominator, the factor being the sign of the permutation that rearranges $\tau(\mathcal{F})$ to δ. But then again by Theorem 1.1 we derive that when $\beta \neq \delta - \alpha$ all these terms will add up to zero, and when $\beta = \delta - \alpha$ the only surviving term will be one producing the sign $(-1)^{\beta_1 + \beta_2 + \cdots + \beta_n}$ which is precisely the factor we need to produce a 1 in the first case of 2.3. Thus 2.3 holds true as desired and our proof is complete.

It should be of interest to see how the identity in 2.3 may be proved analytically. To this end we need the following auxiliary result which is interesting in its own right.

Proposition 2.1

For a, n integers we have

$$\sum_{i=1}^{n} \frac{x_i^a}{\prod_{\substack{j=1 \\ j \neq i}}^{n} (x_i - x_j)} \;=\; \begin{cases} 0 & \text{if } 0 \leq a < n-1 \, , \\ 1 & \text{if } a = n-1 \, . \end{cases} \tag{2.6}$$

Proof

The Lagrange interpolation formula gives that for a polynomial $P(x)$ of degree $< n$ we have

$$P(x) \;=\; \sum_{i=1}^{n} P(x_i)\, \frac{\prod_{\substack{j=1 \\ j\neq i}}^{n}(x-x_j)}{\prod_{\substack{j=1 \\ j\neq i}}^{n}(x_i-x_j)}\,. \tag{2.7}$$

Dividing by the product $\prod_{j=1}^{n}(x-x_j)$ this may be rewritten as

$$\frac{P(x)}{\prod_{j=1}^{n}(x-x_j)} \;=\; \sum_{i=1}^{n} \frac{P(x_i)}{\prod_{\substack{j=1 \\ j\neq i}}^{n}(x_i-x_j)}\; \frac{1}{x-x_i}\,. \tag{2.8}$$

Setting $P(x)=x^a$ (for $a\le n-1$) and replacing x by $1/t$ gives

$$\frac{t^{n-1-a}}{\prod_{j=1}^{n}(1-t\,x_j)} \;=\; \sum_{i=1}^{n} \frac{x_i^a}{\prod_{\substack{j=1 \\ j\neq i}}^{n}(x_i-x_j)}\; \frac{1}{1-t\,x_i}\,.$$

This given setting $t=0$ we obtain 2.6 precisely as asserted.

The analytic proof that $\langle x^\alpha\,,\,\phi_\beta\rangle=0$ when $|\alpha|+|\beta|\neq\binom{n}{2}$ is based on the same idea that yielded part (a) and (b) of Theorem 1.1. So we shall not repeat ourselves here. This given we shall prove 2.3 under the assumption that

$$|\alpha|+|\beta|=\binom{n}{2}. \tag{2.9}$$

To begin note that we may also compute the scalar product defined in 2.1 according to the formula

$$\langle f\,,\,g\rangle \;=\; \sum_{\alpha\in S_n} \alpha\Big(\frac{f(x)\,g(x)}{\prod_{n\geq i>j\geq 1}(x_i-x_j)}\Big) \tag{2.10}$$

Now for $f(x)=x^\alpha$ and $g(x)=\phi_\beta(x)$ we may use the decomposition of S_n into left cosets of S_{n-1} and write their scalar product in the form

$$\langle x^\alpha\,,\,\phi_\beta(x)\rangle=(-1)^{|\beta|}\sum_{i=1}^{n}(i,n)\Big(\frac{x_n^{\alpha_n}e_{\beta_n}(X_{n-1})}{(x_n-x_1)(x_n-x_2)\cdots(x_n-x_{n-1})}\sum_{\alpha\in S_{n-1}}\alpha\Big(\frac{x_1^{\alpha_1}\cdots x_{n-1}^{\alpha_{n-1}}e_{\beta_2}(X_1)\cdots e_{\beta_{n-1}}(X_{n-2})}{\prod_{n-1\geq i>j\geq 1}(x_i-x_j)}\Big)\Big)$$

Thus if we inductively assume that 2.3 has been proved for $n-1$ variables, then the inner summand will not vanish only when

$$\beta_i+\alpha_i=i-1 \qquad \text{for} \quad i=1,2,\ldots,n-1\,. \tag{2.11}$$

But then the assumption in 2.9 gives that we must also have

$$\alpha_n+\beta_n=n-1 \tag{2.12}$$

That proves the second case of 2.3. But when 2.11 and 2.12 hold true, the inductive hypothesis gives that

$$\begin{aligned} \big\langle x^\alpha \ , \ \phi_\beta(x)\big\rangle &= (-1)^{\beta_n}\sum_{i=1}^n (i,n)\bigg(\frac{x_n^{\alpha_n} e_{\beta_n}(X_{n-1})}{(x_n-x_1)(x_n-x_2)\cdots(x_n-x_{n-1})}\bigg) \\ &= (-1)^{\beta_n}\sum_{i=1}^n \frac{x_i^{\alpha_n} e_{\beta_n}(X_n-x_i)}{\prod_{\substack{j=1\\ j\neq i}}^n (x_i-x_j)}\,. \end{aligned} \tag{2.13}$$

Using the addition formula

$$e_{\beta_n}(X_n-x_i) \;=\; \sum_{r=0}^{\beta_n}(-x_i)^r\, e_{\beta_n-r}(X_n)$$

the identity in 2.13 yields

$$\big\langle x^\alpha \ , \ \phi_\beta(x)\big\rangle \;=\; (-1)^{\beta_n}\sum_{r=0}^{\beta_n}(-1)^r\, e_{\beta_n-r}(X_n)\sum_{i=1}^n \frac{x_i^{\alpha_n+r}}{\prod_{\substack{j=1\\ j\neq i}}^n (x_i-x_j)}\,. \tag{2.14}$$

We can now use Proposition 2.1 together with 2.12 to derive that the inner sum vanishes for $r<\beta_n$ reducing 2.14 to

$$\big\langle x^\alpha \ , \ \phi_\beta(x)\big\rangle = (-1)^{\beta_n}(-1)^{\beta_n}\sum_{i=1}^n \frac{x_i^{\alpha_n+\beta_n}}{\prod_{\substack{j=1\\ j\neq i}}^n (x_i-x_j)} = \sum_{i=1}^n \frac{x_i^{n-1}}{\prod_{\substack{j=1\\ j\neq i}}^n (x_i-x_j)} = 1$$

proving the first case of 2.3 and completing the induction argument. We must still show that the induction can be started. We can do this by showing that 2.3 is true for $n=2$. In this case we have only two subtriangular vectors, namely

$$(0,0) \quad \text{and} \quad (0,1)$$

So the two possibilities for x^α are

$$1 \quad \text{and} \quad x_2\,.$$

On the other hand 2.2 gives

$$\phi_{0,0}=1 \quad \text{and} \quad \phi_{0,1}=-x_1$$

Using 2.10 we see that

$$\big\langle 1 \ , \ 1\big\rangle = \big\langle x_2 \ , \ \phi_{0,1}\big\rangle = 0$$

and

$$\big\langle x_2 \ , \ 1\big\rangle = \frac{x_2-x_1}{x_2-x_1} = 1 \quad \text{and} \quad \big\langle 1 \ , \ \phi_{0,1}\big\rangle = -\frac{x_1-x_2}{x_2-x_1} = 1\,.$$

Thus 2.3 holds true in this case and our argument is complete.

3. Expansions in $\mathbf{Q}[\mathbf{x_1},\mathbf{x_2},\ldots,\mathbf{x_n}]$

Let $\mathcal{P}_n$ denote the module of polynomials in $x_1,x_2,\ldots,x_n$ with integer coefficients and Λ_n denote the submodule of symmetric polynomials in $\mathcal{P}_n$. In symbols

$$\mathcal{P}_n = \mathbb{N}[x_1,x_2,\ldots,x_n]$$

$$\Lambda_n = \mathbb{N}[x_1,x_2,\ldots,x_n]^{S_n}$$

From the fundamental Theorem of the Theory of Symmetric Functions we derive that every polynomial $P\in\Lambda_n$ has a unique expansion of the form

$$P \;=\; \sum_{p\geq 0} c_p\, e_1^{p_1}[X_n]e_2^{p_2}[X_n]\cdots e_n^{p_n}[X_n]$$

with coefficients $c_p\in\mathbb{N}$. Theorem 2.1 allows us to extend this result to $\mathcal{P}_n$ in a rather remarkable way. To be precise we shall show here as a first application of our pebbles' magics that

Theorem 3.1

Every polynomial $P\in P_n$ has a unique expansion of the form

$$P \;=\; \sum_{\alpha\leq\delta}\sum_{p\geq 0} c_{\alpha,p}\, x^{\alpha}\, e_1^{p_1}[X_n]e_2^{p_2}[X_n]\cdots e_n^{p_n}[X_n] \qquad (\textit{with } c_{\alpha,p}\in\mathbb{N}) \tag{3.1}$$

where $\alpha\leq\delta$ means that $0\leq\alpha_i\leq i-1$ for $i=1,2,\ldots,n$ and $p\geq 0$ means that $p_i\geq 0$ for $i=1,2,\ldots,n$. In other words we may write

$$P \;=\; \sum_{\alpha\leq\delta} A_\alpha[X_n]\, x^{\alpha} \tag{3.2}$$

with the coefficients $A_\alpha[X_n]$ uniquely determined elements of Λ_n. In fact they may be simply computed from the formula

$$A_\alpha[X_n] \;=\; \big\langle P\,,\,\phi_{\delta-\alpha}\big\rangle \tag{3.3}$$

Proof

The existence and uniqueness of the expansion in 3.1 for every $P\in\mathbf{Q}[x_1,x_2,\ldots,x_n]$ is equivalent to the statement that the collection of polynomials

$$\mathcal{C}_n \;=\; \Big\{x^{\alpha}e_1^{p_1}[X_n]e_2^{p_2}[X_n]\cdots e_n^{p_n}[X_n]\,:\,\alpha\leq\delta\;\;\&\;\;p\geq 0\,\Big\} \tag{3.4}$$

is a basis for $\mathbf{Q}[x_1,x_2,\ldots,x_n]$. However, to show that it is a basis we need only show independence. Indeed, it is easily seen that

$$\sum_{P\in\mathcal{C}_n} q^{degree(P)} \;=\; \frac{\prod_{i=1}^n(1+q+\cdots+q^{i-1})}{(1-q)(1-q^2)\cdots(1-q^n)} \;=\; \frac{1}{(1-q)^n}\,. \tag{3.5}$$

This means that the number of elements of $\mathcal{C}_n$ of any given degree is the same as the number of monomials in $x_1, x_2, \ldots, x_n$ of that same degree. This may also be rewritten as

$$\sum_{P \in \mathcal{C}_n} q^{degree(P)} \;=\; \sum_{d \geq 0} q^d \, dim \mathcal{H}_d\big(\mathbf{Q}[x_1, x_2, \ldots, x_n]\big). \qquad 3.5$$

where $\mathcal{H}_d\big(\mathbf{Q}[x_1, x_2, \ldots, x_n]\big)$ denotes the subspace of homogeneous elements of degree d in $\mathbf{Q}[x_1, x_2, \ldots, x_n]$. So $\mathcal{C}_n$ has the correct number of elements in each subspace $\mathcal{H}_d\big(\mathbf{Q}[x_1, x_2, \ldots, x_n]\big)$, which means that if they are independent they must also span the subspace. This given, assume if possible that we have a collection of symmetric polynomials $\{A_\alpha[X_n]\}_{\alpha \leq \delta}$ such that

$$0 \;=\; \sum_{\alpha \leq \delta} x^\alpha A_\alpha[X_n]\,. \qquad 3.6$$

Now from the formula in 2.10 it immediately follows that if A is symmetric then

$$\big\langle Af \;,\; g \big\rangle \;=\; \big\langle f \;,\; Ag \big\rangle \;=\; A \big\langle f \;,\; g \big\rangle\,.$$

Thus taking the scalar product of 3.6 by ϕ_β and using formula 2.3 we derive that

$$0 \;=\; \sum_{\alpha \leq \delta} A_\alpha \big\langle x^\alpha \;,\; \phi_\beta \big\rangle \;=\; A_{\delta - \beta} \qquad (\text{ for all } \beta \leq \delta\ )$$

This proves independence. So the expansions in 3.1 do exist and are unique for every $P \in \mathbf{Q}[x_1, x_2, \ldots, x_n]$.

Formula 3.3 is proved in exactly the same way. Indeed if

$$P \;=\; \sum_{\alpha \leq \delta} A_\alpha[X_n]\, x^\alpha$$

then taking the scalar product by ϕ_β gives

$$\big\langle P \;,\; \phi_\beta \big\rangle \;=\; \sum_{\alpha \leq \delta} A_\alpha \big\langle x^\alpha \;,\; \phi_\beta \big\rangle \;=\; A_{\delta - \beta}\;.$$

Finally note that, given 2.3, the definitions in 2.1 and 2.2 immediately imply that if $P \in \mathcal{P}_n$ then $\langle P \;,\; \phi_\beta \rangle \in \Lambda_n$. This completes our proof.

Remark 3.1

We should mention that a similar result may also be obtained for the collection $\{\phi_{\delta-\alpha}\}_{\alpha\leq\delta}$. In fact, from 3.2 and 3.3 with $P = \phi_{\delta-\beta}$ we derive that

$$\phi_{\delta-\beta} \;=\; \sum_{\alpha\leq\delta} \big\langle \phi_{\delta-\beta} \,,\, \phi_{\delta-\alpha} \big\rangle \, x^{\alpha} \,.$$

Taking the scalar product of both sides by x^{γ} gives

$$\big\langle x^{\gamma} \,,\, \phi_{\delta-\beta} \big\rangle \;=\; \sum_{\alpha\leq\delta} \big\langle \phi_{\delta-\beta} \,,\, \phi_{\delta-\alpha} \big\rangle \big\langle x^{\gamma} \,,\, x^{\alpha} \big\rangle \,.$$

Since both matrices $\|\langle \phi_{\delta-\beta} \,,\, \phi_{\delta-\alpha}\rangle\|$ and $\|\langle x^{\gamma} \,,\, x^{\alpha}\rangle\|$ have entries in Λ_n from formula 2.3 we derive that their determinants must be ± 1. This implies that they both have inverses with entries in Λ_n. In particular we must have the expansions

$$x^{\alpha} \;=\; \sum_{\beta\leq\delta} \phi_{\delta-\beta} \, u_{\beta,\alpha}[X_n]$$

with $u_{\beta,\alpha}[X_n] \in \Lambda_n$. Using this in 3.2 we derive that every polynomial $P \in \mathcal{P}_n$ may be expanded in the form

$$P \;=\; \sum_{\beta\leq\delta} \phi_{\delta-\beta} \, B_{\beta}[X_n] \qquad 3.7$$

with $B_{\beta}[X_n] \in \Lambda_n$. This given, the equations in 2.3 yield that

$$B_{\beta}[X_n] \;=\; \big\langle P \,,\, x^{\beta} \big\rangle \,. \qquad 3.8$$

This shows that also the expansion in 3.7 is unique.

Remark 3.2

In contemporary jargon Theorem 3.1 proves that $\mathcal{P}_n$ is a free Λ_n-module of rank $n!$ with basis $\{x^{\alpha} : \alpha \leq \delta\}$. The beauty of the pebble's magic is that it yields a purely combinatorial way of proving an explicit formula for coefficients in the resulting expansions.

Let us recall that two bases $\{a_i\}_{i=1}^{n}$ and $\{\beta_i\}_{i=1}^{n}$ are said to be "*dual*" with respect to a given scalar product "(,)" if $\|(a_i \,,\, b_j)\|_{i,j=1}^{n}$ is the identity matrix. For such pairs the expression

$$\sum_{i=1}^{n} a_i \otimes b_i$$

is usually called the "*Reproducing Kernel*". Theorem 2.1 essentially states that the bases $\{x^{\alpha}\}_{\alpha\leq\delta}$ and $\{\phi_{\delta-\alpha}\}_{\alpha\leq\delta}$ are dual with respect to the scalar product "$\langle \;,\; \rangle$". Thus a reproducing kernel for our scalar product is given by the expression

$$\Delta_n(x,y) \;=\; \sum_{\alpha\leq\delta} x^{\alpha} \phi_{\delta-\alpha}(y) \,. \qquad 3.9$$

We are thus led to the following beautiful identities.

Theorem 3.2

$$\Delta(x,y) \;=\; \prod_{n\ge i>j\ge 1} (x_i - y_j)\,. \qquad 3.10$$

Moreover for every polynomial $P\in\mathcal{P}_n$ we have

$$P(x) \;=\; \big\langle P(y)\ ,\ \Delta(x,y)\big\rangle_y \Big|_{y_i\to x_i} \qquad 3.11$$

as well as

$$P(y) \;=\; \big\langle P(x)\ ,\ \Delta(x,y)\big\rangle_x \Big|_{x_i\to y_i} \qquad 3.12$$

where "$\langle\ ,\ \rangle_y$" and "$\langle\ ,\ \rangle_x$" represent taking the scalar product with respect to the y and x-variables respectively.

Proof

Note that

$$\begin{aligned}
\prod_{n\ge i>j\ge 1} (x_i - y_j) \;&=\; \prod_{i=2}^{n} (x_i-y_1)(x_i-y_2)\cdots(x_i-y_{i-1}) \\
&=\; \prod_{i=2}^{n}\Big(\sum_{\alpha_i=0}^{i-1} x_i^{\alpha_i}(-1)^{i-1-\alpha_i} e_{i-1-\alpha_i}[Y_{i-1}]\Big)\,. \\
\text{(by 2.2)} \;&=\; \sum_{\alpha\le\delta} x^{\alpha}\,\phi_{\delta-\alpha}(y)
\end{aligned}$$

This proves 3.10.

Next we note that combining 3.2 and 3.3 we derive that every $P\in\mathcal{P}_n$ has an expansion of the form

$$P(x) \;=\; \sum_{\alpha\le\delta} a_\alpha(x)\,\big\langle P(y)\ ,\ \phi_{\delta-\alpha}(y)\big\rangle_y \Big|_{y_i\to x_i}$$

and from 3.9 we derive that this is just another way of writing 3.11. Similarly combining 3.7 and 3.8 we derive the expansion

$$P(y) \;=\; \sum_{\beta\le\delta} \phi_{\delta-\beta}(y)\,\big\langle P(x)\ ,\ x^{\beta}\rangle_x \Big|_{x_i\to y_i}$$

and this is another way of writing 3.12.

Remark 3.3

It develops that, for a given index set Ξ_n of cardinalty $n!$ we may have a pair of bases

$$\{a_\alpha(x)\}_{\alpha\in\Xi_n} \qquad \text{and} \qquad \{b_\alpha(x)\}_{\alpha\in\Xi_n}\,,$$

dual with respect to the scalar product "$\langle\ ,\ \rangle$" for which the corresponding reproducing kernel

$$\sum_{\alpha\in\Xi_n} a_\alpha[x_1,x_2,\ldots,x_n]\ b_\alpha[y_1,y_2,\ldots,y_n]$$

does not evaluate to $\Delta_n(x,y)$. This is in stark contrast with what happens for standard scalar products, where the reproducing kernel is unique. The basic reason for this is that our scalar product may take "non scalar" values. This circumstance invalidates the customary uniqueness proof. An example in point is obtained when we take $\Xi_n = S_n$ and for $\alpha \in S_n$ we let $a_\alpha(x)$ be the so called "descent monomial":

$$m_\alpha(x) \;\; = \prod_{\alpha_i > \alpha_{i+1}} x_{\alpha_1} x_{\alpha_2} \cdots x_{\alpha_i}$$

In fact, it was shown in [1] that for every $P \in \mathbf{Q}[x_1, x_2, \ldots, x_n]$ we have a unique expansion of the form

$$P(x) \;\; = \sum_{\alpha \in S_n} A_\alpha(x)\, m_\alpha(x)$$

with the coefficients $A_\alpha(x)$in Λ_n. Thus also $\{m_\alpha(x)\}_{\alpha\in S_n}$ is a Λ_n-module basis for $\mathcal{P}_n$. It can be shown that the Gramm matrix of $\{m_\alpha(x)\}_{\alpha\in S_n}$ with respect to "$\langle \; , \; \rangle$" is uni-triangular under suitable total orders of S_n and thus may be inverted to yield a polynomial dual basis $\{d_\alpha(x)\}_{\alpha\in S_n}$. However, note that for $n=3$ we obtain

$$m_{123}(x) = 1\ , \quad m_{132}(x) = x_1x_3\ , \quad m_{213}(x) = x_2\ , \quad m_{231}(x) = x_2x_3\ , \quad m_{312}(x) = x_3\ , \quad m_{321}(x) = x_3^2 x_2\ ,$$

and it is easily seen that there are no choices of $\{d_\alpha\}_{\alpha\in S_3}$ giving

$$\sum_{\alpha\in S_3} m_\alpha(x)\, d_\alpha(y) \;\; = \;\; (x_2-y_1)(x_3-y_1)(x_3-y_2)\,.$$

We must mention that this particular observation is due to F. Bergeron and C. Reutenauer. [(†)]. It should be of interest to find an explicit formula for the reproducing kernel of the descent basis and its dual.

The above example notwithstanding, we can nevertheless obtain a uniqueness result even for the scalar product "$\langle \; , \; \rangle$". To be precise we have

Theorem 3.3

Let Ξ_n be an index set of cardinalty $n!$ and let

$$\{a_\alpha(x)\}_{\alpha\in\Xi_n}\ , \qquad \{b_\alpha(x)\}_{\alpha\in\Xi_n} \tag{3.13}$$

be two collections of polynomials in $\mathcal{P}_n$ satisfying for $\alpha,\beta\in\Xi_n$ the duality condition

$$\langle a_\alpha \ , \ b_\beta \rangle \;\; = \;\; \begin{cases} 1 & \text{if } \alpha=\beta \\ 0 & \text{if } \alpha\neq\beta \end{cases} \tag{3.14}$$

Moreover assume that for some integers $\{c_{\epsilon,\alpha}\}_{\epsilon\le\delta\,,\alpha\in\Xi_n}$ we have the expansions

$$a_\alpha(x) \;\; = \sum_{\epsilon\le\delta} x^\epsilon c_{\epsilon,\alpha} \qquad (\forall\, \alpha\in\Xi_n\ ) \tag{3.15}$$

(†) Personal communication

then the collections in 3.13 are both Λ_n-module bases for $\mathcal{P}_n$ and

$$\sum_{\alpha\in\Xi_n} a_\alpha[x_1,x_2,\ldots,x_n]b_\alpha[y_1,y_2,\ldots,y_n] \;=\; \prod_{n\geq i>j\geq 1}(x_i-y_j)\,. \qquad 3.16$$

Proof

Taking the scalar product of 3.15 with $b_\beta(x)$ gives

$$\big\langle a_\alpha\,,\,b_\beta\big\rangle \;=\; \sum_{\epsilon\leq\delta}\big\langle x^\epsilon\,,\,b_\beta\big\rangle\,c_{\epsilon,\alpha}\,.$$

Thus 3.14 implies that the matrix $\|c_{\epsilon,\alpha}\|_{\epsilon\neq\delta,\alpha\in\Xi_n}$ has determinant ± 1, and is therefore invertible over the integers. Denoting by $d_{\epsilon,\alpha}$ the entries of its inverse we obtain the expansions

$$x^\epsilon \;=\; \sum_{\alpha\in\Xi_n} a_\alpha(x)\,d_{\epsilon,\alpha} \qquad (\,\forall\,\epsilon\leq\delta\,)\,. \qquad 3.17$$

Using this in 3.9 we obtain

$$\begin{aligned}\Delta_n(x,y) \;&=\; \sum_{\epsilon\leq\delta}\Big(\sum_{\alpha\in\Xi_n} a_\alpha(x)\,d_{\epsilon,\alpha}\Big)\phi_{\delta-\epsilon}(y)\\ &=\; \sum_{\alpha\in\Xi_n} a_\alpha(x)\Big(\sum_{\epsilon\leq\delta} d_{\epsilon,\alpha}\phi_{\delta-\epsilon}(y)\Big)\end{aligned} \qquad 3.18$$

Taking the $\langle\,,\,\rangle_x$ scalar product of both sides by $b_\beta(y)$, using 3.12 on the left hand side and 3.14 on the right hand side we obtain that

$$b_\beta(y) \;=\; \sum_{\epsilon\leq\delta} d_{\epsilon,\alpha}\phi_{\delta-\epsilon}(y)$$

Substituting this back into 3.18 and using 3.10 gives 3.16. But now 3.16 allows us to we rewrite 3.11 and 3.12 respectively in the form

$$P(x)=\sum_{\alpha\leq\delta} a_\alpha(x)\,\big\langle P(y)\,,\,b_\alpha(y)\big\rangle_y\Big|_{y_i\to x_i} \qquad \text{and} \qquad P(x)=\sum_{\beta\leq\delta} b_\beta(y)\,\big\langle P(x)\,,\,a_\beta(x)\big\rangle_x\Big|_{x_i\to y_i}$$

This shows that $\{a_\alpha(x)\}_{\alpha\in\Xi_n}$ and $\{b_\alpha(x)\}_{\alpha\in\Xi_n}$ are Λ_n-bases for $\mathcal{P}_n$ because the unicity in these expansions follows immediately from the identities in 3.14.

Lascoux-Schützenberger in [2] (see also [3]) introduced the scalar product $\langle\,,\,\rangle$ as a natural ingredient in the study of "*Schubert Polynomials*". In fact, they proved that these polynomials are essentially the self dual basis for this scalar product. What we have established in these note allows us to derive their result with very little more effort. It is therefore worthwhile that this further consequence of pebble magics should also be included. To this end we need to recall some definitions. The basic ingredients in this context are the "*divided difference*" operators δ_i of Lascoux-Schützenberger. These are defined by setting for every polynomial $P = P(x_1, x_2, \ldots, x_n)$

$$\delta_i P \;=\; \frac{1}{x_i - x_{i+1}}(P - s_i P) \qquad \text{(for } i = 1, 2, \ldots n-1\text{)} \tag{3.19}$$

where s_i denotes the simple transposition

$$s_i = (i, i+1)\,.$$

We recall that $(i, i+1)$ acts on a polynomial in $x_1, x_2, \ldots, x_n$ by interchanging x_i and x_{i+1}. Note that for any monomial $x^p = x_1^{p_1} x_2^{p_2} \cdots x_n^{p_n}$ we have

$$\begin{aligned} \delta_i x^p \;&=\; x_1^{p_1} x_2^{p_2} \cdots x_{i-1}^{p_{i-1}} \Big(\frac{x_i^{p_i} - x_{i+1}^{p_i}}{x_i - x_{i+1}}\Big) x_{i+2}^{p_{i+2}} \cdots x_n^{p_n} \\ &=\; \sum_{j=0}^{p_i - 1} x_1^{p_1} x_2^{p_2} \cdots x_{i-1}^{p_{i-1}} x_i^{p_i - 1 - j} x_{i+1}^{j} x_{i+2}^{p_{i+2}} \cdots x_n^{p_n} \end{aligned} \tag{3.20}$$

We see from this identity that δ_i sends any polynomial into a polynomial that is symmetric in x_i, x_{i+1}. Moreover if P is symmetric in x_i, x_{i+1}, then $P - s_i P = 0$ and 3.19 then gives $\delta_i P = 0$. In particular we deduce that

$$\delta_i^2 \;=\; 0 \tag{3.21}$$

Note that δ_i has a Leibnitz formula. Indeed note that the definition in 3.16 gives

$$\delta_i PQ \;=\; \frac{(P - s_i P)Q}{x_i - x_{i+1}} + (s_i P)\frac{(Q - s_i Q)}{x_i - x_{i+1}}\,.$$

Since $s_i \delta_i Q = \delta_i Q$, this may be rewritten as

$$\delta_i PQ \;=\; (\delta_i P)Q \;+\; s_i(P \delta_i Q)\,, \tag{3.22}$$

Another important identity satisfied by the difference operators δ_i is the "*Coxeter relation*"

$$\delta_i\, \delta_{i+1} \delta_i \;=\; \delta_{i+1} \delta_i\, \delta_{i+1}\,. \tag{3.23}$$

We recall that a factorization of a permutation σ into a product of simple reflections

$$\sigma \;=\; s_{i_1} s_{i_2} \cdots s_{i_k} \tag{3.24}$$

is called "*reduced*" if the number of factors is as small as possible. In other words 3.24 is reduced if and only if k is equal to the number of inversions of σ. The latter is usually called the "'*length*" of σ and is denoted $l(\sigma)$.

We may now define a difference operator "δ_σ" by setting

$$\delta_\sigma \;=\; \delta_{i_1} \delta_{i_2} \cdots \delta_{i_k}\,. \tag{3.25}$$

In fact it follows from 3.23 that the expression on the right hand side of 3.25 depends only on σ and not on the particular factorization that is used, as long as it is reduced. We should point out that it follows from 3.21 that the product on the right hand side of 3.25 will vanish if 3.24 is not reduced.

It will be convenient to denote by $\sigma^{(n)}$ the top permutation of S_n. That is

$$\sigma^{(n)} \;=\; \begin{bmatrix} 1 & 2 & 3 & \cdots & n-1 & n \\ n & n-1 & n-2 & \cdots & 2 & 1 \end{bmatrix}$$

This given, the "*Schubert polynomial*" $\mathcal{SH}_\sigma(x)$ is simply defined by setting

$$\mathcal{SH}_\sigma(x) \;=\; \mathcal{SH}_\sigma(x_1, x_2, \ldots, x_n) \;=\; \delta_{\sigma^{-1}\sigma^{(n)}} x_1^{n-1} x_2^{n-2} \cdots x_{n-1} \qquad 3.26$$

It follows from this definition (together with 3.21 and 3.26) that we have

$$\delta_\alpha \mathcal{SH}_\beta(x) \;=\; \begin{cases} \mathcal{SH}_{\beta\alpha^{-1}}(x) & \text{if } l(\beta\alpha^{-1}) = l(\beta) - l(\alpha)\ , \\ 0 & \text{otherwise}\ . \end{cases} \qquad 3.27$$

The relation between the Schubert polynomials and the scalar product $\langle\ ,\ \rangle$ stems from the following two important properties

Proposition 3.1

For any two polynomials P, Q we have

$$\begin{aligned} &a) \quad \langle s_i P\ ,\ Q \rangle \;=\; -\langle P, s_i Q \rangle . \\ &b) \quad \langle \delta_i P\ ,\ Q \rangle \;=\; \langle P\ ,\ \delta_i Q \rangle , \end{aligned} \qquad 3.28$$

Proof

From 2.10 it follows that the scalar product may also be defined by setting

$$\langle P\ ,\ Q \rangle \;=\; \frac{1}{\prod_{n \geq i > j \geq 1} (x_i - x_j)} \mathcal{A}_n\, PQ\,, \qquad 3.29$$

where for a polynomial P we set

$$\mathcal{A}_n\, P \;=\; \sum_{\alpha \in S_n} sign(\sigma) \sigma\, P\,.$$

It is easily seen that for any $i = 1, \ldots n-1$ we have

$$\mathcal{A}_n \;=\; -\mathcal{A}_n s_i\,.$$

Thus

$$\mathcal{A}_n(s_i P) Q \;=\; -\mathcal{A}_n\, P s_i Q\,.$$

Dividing by $\prod_{n \geq i > j \geq 1} (x_i - x_j)$ gives 3.28 a).

To prove 3.28 b) note that if P is symmetric in x_i, x_{i+1} then $\mathcal{A}_n P = \mathcal{A}_n\, s_i P = -\mathcal{A}_n P$, and this forces $\mathcal{A}_n P = 0$. In particular we must also have

$$\mathcal{A}_n \delta_i (PQ) \;=\; 0\,.$$

But now from 3.22 we get

$$0 \;=\; \mathcal{A}_n \delta_i (PQ) \;=\; \mathcal{A}_n (\delta_i P) Q \;+\; \mathcal{A}_n\, s_i (P \delta_i Q) \;=\; \mathcal{A}_n (\delta_i P) Q \;-\; \mathcal{A}_n\, P \delta_i Q$$

dividing by $\prod_{n \geq i > j \geq 1} (x_i - x_j)$ and using 3.29 gives 3.28 b). This completes our proof.

An immediate consequence of this proposition is the following beatiful result of Lascoux-Schützenberger:

Theorem 3.4

For $\alpha, \beta \in S_n$ we have

$$\left\langle \sigma^{(n)} \mathcal{SH}_\alpha \,,\, \mathcal{SH}_{\beta \sigma^{(n)}} \right\rangle \;=\; \begin{cases} (-1)^{\binom{n}{2}} sign(\alpha) & \text{if } \alpha = \beta \\ 0 & \text{otherwise} \end{cases} \qquad 3.30$$

Proof

Recalling that we have set $\delta = (0, 1, 2, \ldots, n-1)$, we may simply write

$$x_1^{n-1} x_2^{n-2} \cdots x_{n-1} \;=\; \sigma^{(n)} x^\delta\,.$$

This given, we have

$$\left\langle \sigma^{(n)} \mathcal{SH}_\alpha \,,\, \mathcal{SH}_{\beta \sigma^{(n)}} \right\rangle \;=\; \left\langle \sigma^{(n)} \delta_{\alpha^{-1} \sigma^{(n)}} \sigma^{(n)} x^\delta \,,\, \mathcal{SH}_{\beta \sigma^{(n)}} \right\rangle \qquad 3.31$$

Now it is easily verified that we have the relation

$$\sigma^{(n)} \delta_i\, \sigma^{(n)} \;=\; -\delta_{n-i}$$

and this implies that

$$\sigma^{(n)} \delta_\sigma\, \sigma^{(n)} \;=\; sign(\sigma) \delta_{\sigma^{(n)} \sigma\, \sigma^{(n)}}$$

Taking account that $sign\big(\alpha^{-1} \sigma^{(n)}\big) = (-1)^{\binom{n}{2}} sign(\alpha)$ we derive that

$$\sigma^{(n)} \delta_{\alpha^{-1} \sigma^{(n)}} \sigma^{(n)} \;=\; (-1)^{\binom{n}{2}} sign(\alpha)\, \delta_{\sigma^{(n)} \alpha^{-1}}\,.$$

Substituting this in 3.31 gives, using 3.28 b)

$$\begin{aligned} \left\langle \sigma^{(n)} \mathcal{SH}_\alpha \,,\, \mathcal{SH}_{\beta \sigma^{(n)}} \right\rangle \;&=\; (-1)^{\binom{n}{2}} sign(\alpha) \left\langle \delta_{\sigma^{(n)} \alpha^{-1}}\, x^\delta \,,\, \mathcal{SH}_{\beta \sigma^{(n)}} \right\rangle \\ &=\; (-1)^{\binom{n}{2}} sign(\alpha) \left\langle x^\delta \,,\, \delta_{\alpha\, \sigma^{(n)}}\, \mathcal{SH}_{\beta \sigma^{(n)}} \right\rangle \end{aligned} \qquad 3.32$$

But now, 3.27 gives

$$\delta_{\alpha\,\sigma^{(n)}}\,\mathcal{SH}_{\beta\sigma^{(n)}} \;=\; \begin{cases} \mathcal{SH}_{\beta\alpha^{-1}} & \text{if } l(\beta\alpha^{-1}) = l(\beta\,\sigma^{(n)}) - l(\alpha\,\sigma^{(n)}), \\ 0 & \text{otherwise.} \end{cases}$$

Since $l(\alpha\,\sigma^{(n)}) = \binom{n}{2} - l(\alpha)$ and $l(\beta\,\sigma^{(n)}) = \binom{n}{2} - l(\beta)$, 3.32 reduces to

$$\left\langle \sigma^{(n)}\mathcal{SH}_\alpha \;,\; \mathcal{SH}_{\beta\sigma^{(n)}} \right\rangle \;=\; sign(\alpha) \begin{cases} (-1)^{\binom{n}{2}} \left\langle x^\delta \;,\; \mathcal{SH}_{\beta\alpha^{-1}} \right\rangle & \text{if } l(\beta\alpha^{-1}) = l(\alpha) - l(\beta), \\ 0 & \text{otherwise.} \end{cases} \tag{3.33}$$

Finally note that from 3.20 and the definition in 3.26 it follows that the polynomial $\mathcal{SH}_{\beta\alpha^{-1}}$ is necessarily of the form

$$\mathcal{SH}_{\beta\alpha^{-1}} = \sum_{p \le \sigma^{(n)}\delta} c_p x^p \tag{3.34}$$

where the symbol "$p \le \sigma^{(n)}\delta$" is to express that the sum is over exponent vectors $p = (p_1, p_2, \ldots, p_n)$ with

$$p_1 \le n-1 \;,\; p_2 \le n-2, \ldots, p_{n-1} \le 1 \;,\; p_n = 0\,. \tag{3.35}$$

Taking this into account we get, using 3.29

$$\left\langle x^\delta \;,\; \mathcal{SH}_{\beta\alpha^{-1}} \right\rangle \;=\; \frac{1}{\prod_{n \ge i > j \ge 1}(x_i - x_j)} \sum_{p \le \sigma^{(n)}\delta} c_p\, \mathcal{A}_n x^{\delta+p}\,.$$

Now the term $\mathcal{A}_n x^{\delta+p}$ necesserily vanishes unless the components of $\delta+p$ are all distinct. On the other hand, the inequalities in 3.35 yield that all the compents of $\delta+p$ are necessarly $\le n-1$, thus if they are distinct they can only be a rearrangement of the components of δ. But that is only possible when $p=0$, forcing $\mathcal{SH}_{\beta\alpha^{-1}}$ to be constant. Since the degree of $\mathcal{SH}_{\beta\alpha^{-1}}$, in 3.33, is necessarily $l(\alpha)-l(\beta)$ this can only happen when $\beta = \alpha$. So, in view of 3.33 to prove 3.30 we are left to show that

$$\left\langle x^\delta \;,\; \mathcal{SH}_{id} \right\rangle \;=\; 1$$

Since it is easily seen from 3.29 that

$$\left\langle x^\delta \;,\; 1 \right\rangle \;=\; 1$$

we are reduced to verify that

$$\mathcal{SH}_{id} = 1\,. \tag{3.36}$$

This is easily shown by induction on n. Indeed, by definition we have

$$\mathcal{SH}_{id} = \delta_{\sigma^{(n)}} x_1^{n-1} x_2^{n-2} \cdots x_{n-1}$$

and since we may write

$$\delta_{\sigma^{(n)}} \;=\; \delta_1 \delta_2 \cdots \delta_{n-1} \delta_{\sigma^{(n-1)}}$$

we have, using the inductive hypothesis

$$
\begin{aligned}
\delta_{\sigma^{(n)}} x_1^{n-1} x_2^{n-2} \cdots x_{n-1} &= \delta_1 \delta_2 \cdots \delta_{n-1} \delta_{\sigma^{(n-1)}} x_1^{n-1} x_2^{n-2} \cdots x_{n-1} \\
&= \delta_1 \delta_2 \cdots \delta_{n-1} x_1 x_2 \cdots x_{n-1}\, \delta_{\sigma^{(n-1)}} x_1^{n-2} x_2^{n-3} \cdots x_{n-2} \\
&= \delta_1 \delta_2 \cdots \delta_{n-1} x_1 x_2 \cdots x_{n-1} \\
&= 1
\end{aligned}
$$

This proves 3.35 and completes our proof.

We are finally in a position to state and prove the following remarkable identity.

Theorem 3.5

$$
\sum_{\alpha \in S_n} \mathcal{SH}_\alpha(x)\, \mathcal{SH}_{\alpha\sigma^{(n)}}(y) = \prod_{2 \le i+j \le n} (x_i + y_j) \tag{3.36}
$$

Proof

It follows from 3.30 that the collections $\{(-1)^{\binom{n}{2}} \sigma^{(n)} \mathcal{SH}_\alpha\}_{\sigma \in S_n}$ and $\{sign(\alpha) \mathcal{SH}_{\alpha\sigma^{(n)}}\}_{\sigma \in S_n}$ are dual with respect to the scalar product $\langle\ ,\ \rangle$. From 3.34 and 3.35 we derive that the polynomial $\sigma^{(n)} \mathcal{SH}_\alpha(x)$ has an expansion of the forrm

$$
\sigma^{(n)} \mathcal{SH}_\alpha(x) = \sum_{\epsilon \le \delta} c_\epsilon\, x^\epsilon
$$

Thus we may apply Theorem 3.3 and obtain the identity

$$
\sum_{\alpha \in S_n} (-1)^{\binom{n}{2}} \sigma^{(n)} \mathcal{SH}_\alpha(x)\, sign(\alpha) \mathcal{SH}_{\alpha\sigma^{(n)}}(y) = \prod_{n \ge i > j \ge 1} (x_i - y_j) \tag{3.37}
$$

Since the definition in 3.26 yields that the polynomial $\mathcal{SH}_{\alpha\sigma^{(n)}}$ is homogeneous of degree $l(\alpha\sigma^{(n)}) = l(\binom{n}{2} - \alpha)$ we see that making the replacements $y_i \to -y_i$ and $x_i \to x_{n+1-i}$ changes 3.37 into 3.36 completing the proof of the Theorem.

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