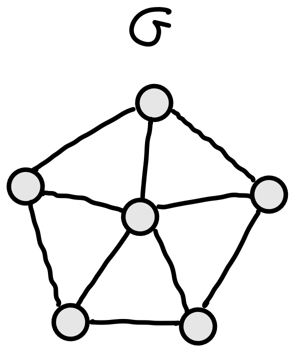
(Proper) coloring of a graph = a way of coloring the vertices so that no two adjacent vertices have the <u>same</u> color.

***k*-coloring** = a coloring that uses exactly k colors.
(If a k-coloring of G exists, say that G is ***k*-colorable**.)

Chromatic number $\chi(G)$ = smallest k so that G is k-colorable.

<u>Warm-up</u>: find $\chi(G)$



Special case with an "easy/nice" order for greedy alg:

<u>Def</u> G is d-degenerate if <u>every</u> <u>subgraph</u> of G has a vtx of degree $\leq d$.

$\delta \leq d \leq \Delta$

Idea: order vtxs so few edges "looking back"

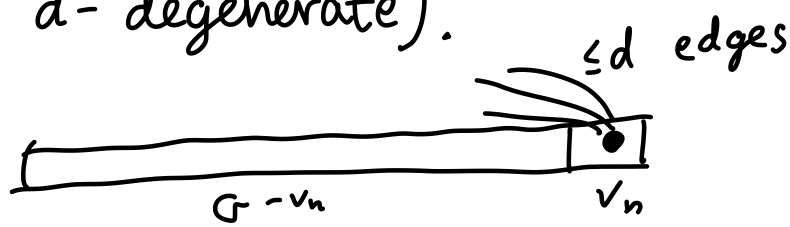
<u>Strategy</u> put low-deg vtxs @ end of ordering.

Prop (6.4.1) If G is d-degenerate, $\chi(G) \le d+1$.

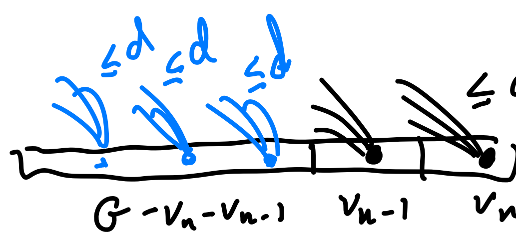
i.e., G is $(d+1)$-colorable

Pf define ordering $v_1, v_2, \ldots, v_n$ of vtxs of G recursively:

Let v_n be a vtx of degree $\le d$ (exists b/c G is d-degenerate).



Delete v_n. $\overbrace{G - v_n}^{\text{subgraph of } G}$ also has ≥ 1 vtx of degree $\le d$. Choose one, call it v_{n-1}.



Recurse, till no vtxs remaining. Now, greedily color G in the order $v_1, v_2, \ldots, v_n$. By construction, each v_i has $\le d$ nbrs among $v_1, v_2, \ldots, v_{i-1}$. So if we have $d+1$ colors, when it's time to color v_i, there's ≥ 1 color available. $\Rightarrow \le d+1$ colors used total. $\square$

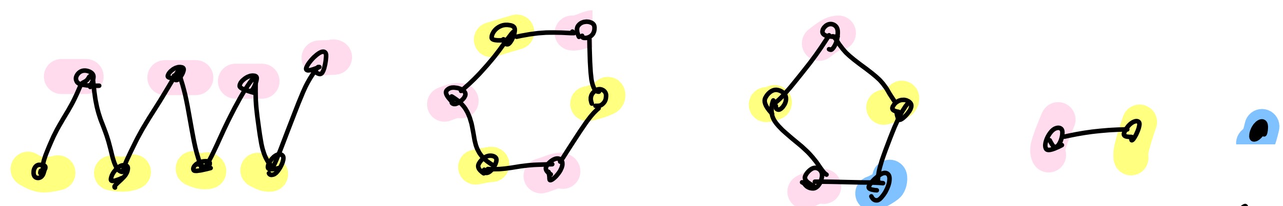
Brooks' Thm Let G be a connected graph with max degree Δ. If G is not a complete graph or an odd cycle, then:

$$\chi(G) \le \Delta.$$

Pf of Brooks' Thm Today + next time.

Easy case $\Delta \le 2$. Then,

Can check: the only connected graphs with $\Delta \le 2$ are paths, cycles, or isolated vtxs.



$\chi = 2,\ \Delta = 2$

even
$\chi = 2,\ \Delta = 2$

odd
$\chi = 3,\ \Delta = 2$

$\chi = 2, \Delta = 1$

$\chi = 1,$
$\Delta = 0$

$\chi = \Delta$

$\chi = \Delta + 1$

odd cycle, so okay

complete graphs, so okay

In all cases $\chi(G) \le \Delta$, unless G is a complete graph or an odd cycle. ✓

Harder case $\Delta \geq 3$.

Let G be a connected graph with max degree $\Delta \geq 3$ that is not a complete graph (i.e., $G \neq K_{\Delta+1}$).

(odd cycle has $\Delta = 2$, so not an option)

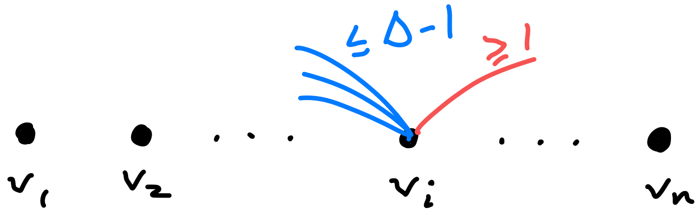
Want to show: $\chi(G) \leq \Delta$.

Idea do greedy coloring, try to save just a little: remember, greedy coloring always uses $\leq \Delta + 1$ colors, so we only need to reduce by one!

i.e., pick 1^{st} available color at each step

Strategy Do greedy coloring on some vtx order $v_1, v_2, \ldots, v_n$. "Dream ordering" would be:

$|V(G)| = n$



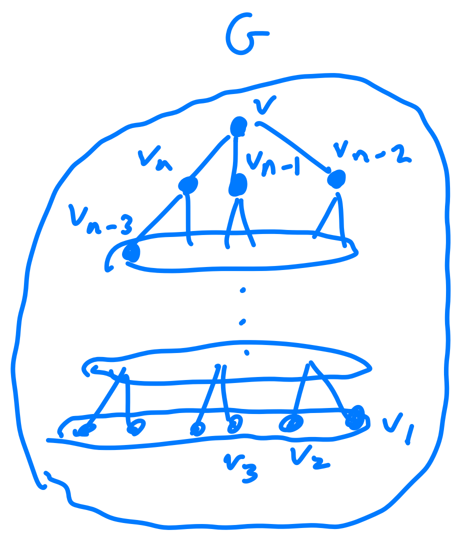
so that each v_i has $\leq \Delta - 1$ neighbors among $v_1, v_2, \ldots, v_{i-1}$ ("backward nbrs").

Then at step i, there is ≥ 1 available color (from among Δ colors) that can be used for v_i.

Surprisingly, this "dream order" is almost possible!

Lemma ☆ For any connected graph G, n vtxs, and any $v \in V(G)$, there is a vtx ordering $v_1, v_2, \ldots, v_n = v$ so that v_i has $\leq d_G(v_i) - 1$ nbrs among $v_1, \ldots, v_{i-1}$, for all i except $i = n$.

Pf idea Do BFS (or DFS) order, with v as the root, then reverse the ordering, to get $\underbrace{v_1}_{\text{last vtx added to tree}}, v_2, \ldots, \underbrace{v_{n-1}}_{\text{2nd vtx added to tree}}, \underbrace{v_n = v}_{\text{root}}$

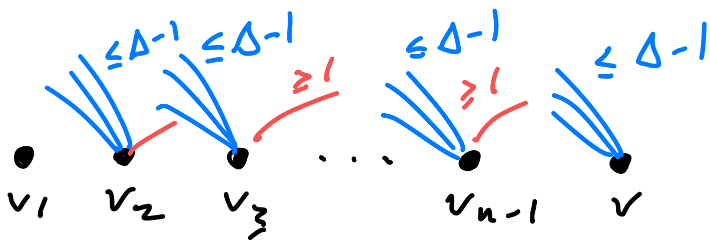


Every vtx v_i other than the root has ≥ 1 "ancestor" in the tree.
$\Rightarrow v_i$ has ≥ 1 nbr later in the ordering, i.e. in $v_{i+1}, \ldots, v_n$.

$\Rightarrow v_i$ has $\leq d_G(v_i) - 1$ nbrs in $v_1, v_2, \ldots, v_{i-1}$.

But what to do about root vtx?

If there is a vtx $v \in V(G)$ with degree $\leq \Delta - 1$, apply lemma ⊛, then do greedy coloring $\Rightarrow$ uses $\leq \Delta$ colors.

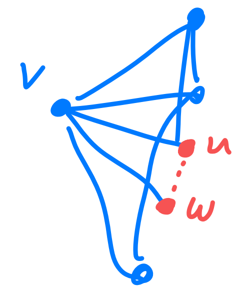


But if <u>every</u> vtx has degree Δ, we're out of luck ☹ Need to "save a color" some other way for last vtx v.

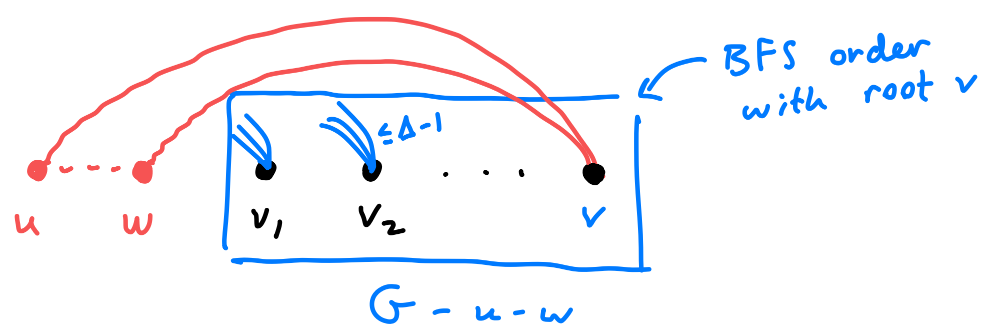
Still haven't used the assumption that $G \neq K_{\Delta+1}$!

Take any vtx v, has degree Δ. Since $G \neq K_{\Delta+1}$, v must have a "missing edge" u,w in its nbhd.

(if every two nbrs of v were adjacent, get $K_{\Delta+1}$.)



<u>Trick</u> apply Lemma ⊛ to $G-u-w$ using v as the root.



Then put u,w at the <u>beginning</u>, and greedily color.

- Since u, w not adjacent, both colored with 1^{st} color.

- For each subsequent vtx before v, $\leq \Delta - 1$ nbrs earlier in ordering, $\Rightarrow \geq 1$ color a ailable.

- And for v, Δ nbrs, but $\leq \Delta - 1$ colors used on nbrs of v (since u, w have <u>same</u> color) $\Rightarrow \geq 1$ color available.

$\Rightarrow$ colored G with $\leq \Delta$ colors! :)

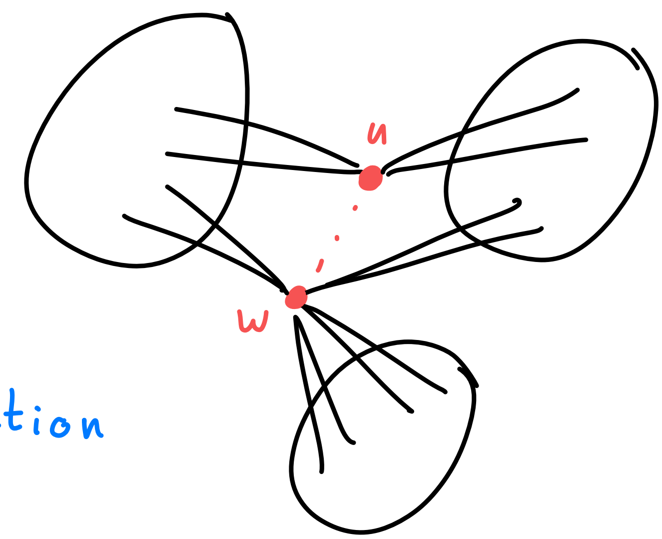
⚠ Not quite done though! There was a subtle flaw in this argument. Can you spot it?

To apply Lemma ✪ to $G-u-w$ (i.e. to build BFS tree), need $G-u-w$ connected!

What if $G-u-w$ is disconnected?

Then G is made of nearly disconnected "chunks". Maybe we can color the chunks separately, then fuse the colorings @ u and w?

Feels like a divide & conquer algorithm! ⇒ use induction



This is what we'll do next time!