Math 180A Homework 1

Winter 2023

Due date: **11:59pm** (Pacific Time) on **Wed Jan 18** (via Gradescope)

For Homework 1 only, you will receive a bonus point if you collaborate with at least 2 other students in our class!

Section 1 (input directly in Gradescope)

Submit the answers to these problems directly through the Gradescope interface. You do not need to write up or explain your work.

Problem 1 (Multiple choice)**.** Alice decides to go to one of the 8 restaurants at the Price Center for lunch. Unbeknownst to her, 13 of her friends are also eating lunch at the Price Center. 2 of the restaurants have 1 friend each, 4 of the restaurants have 2 friends each, and 1 of the restaurants has 3 friends.

1. Suppose that she chooses one out of the 8 restaurants at random. What is the probability that she has at least two friends at the restaurant?
   (a) $\frac{2}{8}$
   (b) $\frac{11}{13}$
   (c) $\frac{5}{8}$
   (d) $\frac{5}{13}$
2. Suppose she calls one of these 13 friends at random in order to meet up for lunch. What is the probability that she called someone at a restaurant where at least two of her friends are present?
   (a) $\frac{2}{8}$
   (b) $\frac{11}{13}$
   (c) $\frac{5}{8}$
   (d) $\frac{5}{13}$

Problem 2 (Multiple choice)**.** Suppose an urn contains seven chips labeled $1, \ldots, 7$. Three of the chips are black, two are red, and two are green. The chips are drawn randomly one at a time without replacement until the urn is empty.

1. What is the probability that the ith draw is chip number 2?

   (a) $\frac{1}{7}$
   (b) $\frac{2}{7}$
   (c) $\frac{i}{7}$
   (d) $\frac{2i}{7}$

2. What is the probability that the ith draw is red?

   (a) $\frac{1}{7}$
   (b) $\frac{2}{7}$
   (c) $\frac{1}{7^i}$
   (d) $\frac{2}{7^i}$

Section 2 (upload files)

For each problem, write your solution on a page by itself, and upload it as a separate file to Gradescope (either typed or scanned from handwritten work). You should write your solutions to these problems neatly and carefully and provide full justification for your answers.

Problem 3. In the California Fantasy 5 Lottery, 5 distinct numbers are picked from 1, ..., 39 uniformly at random. If your ticket matches three out of the five numbers, you will win $15.

(a) Describe a sample space Ω and a probability measure P to model the selection of the five winning numbers.

(b) What is the probability that your ticket matches exactly three out of the five numbers? Give an exact fraction answer.

Problem 4. *(This is Exercise 1.22 from Anderson, Seppäläinen, and Valkó's book.)* We pick a card uniformly at random from a standard deck of 52 cards.

(a) Describe the sample space Ω and the probability measure P that model this experiment.

(b) Give an example of an event in this probability space with probability 3/52.

Problem 5 (Inclusion-exclusion principle)**.**

(a) Draw a Venn diagram of three events A, B, C.

(b) Give a formula for $P(A \cup B \cup C)$ in terms of the probabilities of A, B, C, $A \cap B$, $A \cap C$, $B \cap C$, and $A \cap B \cap C$. You do not need to give a formal proof of the correctness of your formula, but you should write a clear explanation of where it comes from; you may find it helpful to reference the Venn diagram or set operations.

Problem 6. *(This is Exercise 1.14 from Anderson, Seppäläinen, and Valkó's book.)* Assume that $P(A) = 0.4$ and $P(B) = 0.7$. Making no further assumptions on A and B, show that $P(AB)$ satisfies $0.1 \le P(AB) \le 0.4$.

Problem 7. *(This is Exercise 1.40 from Anderson, Seppäläinen, and Valkó's book.)* An urn contains 1 green ball, 1 red ball, 1 yellow ball and 1 white ball. I draw 4 balls with replacement. What is the probability that there is at least one color that is repeated exactly twice?

Hint. Use inclusion-exclusion with events G = {exactly two balls are green}, R = {exactly two balls are red}, etc.

Problem 8. *(This is Exercise 1.43 from Anderson, Seppäläinen, and Valkó's book.)* Show that for any events $A_1, A_2, \ldots, A_n$,

$$P(A_1 \cup \cdots \cup A_n) \le P(A_1) + \cdots + P(A_n) = \sum_{k=1}^{n} P(A_k).$$

Hint: Obtain the case $n = 2$ from inclusion-exclusion. Prove the more general case by using the $n = 2$ case repeatedly, or by mathematical induction.

Problem 9 (Bonus – NOT TO BE TURNED IN)**.** Remember that with 25 people, there is a good chance that two will share the same birthday? This problem will show that out of all the people in human history, the probability that any two of them ever shuffled a deck of cards in the same order is *astronomically small*!

1. A deck of cards has 52 cards which are all distinct. How many possible orderings of the cards are there? Call this number n.

2. Give a ball park estimate of how many times someone has shuffled a deck of cards in human history. Call this number m. Hint: Guess based on the number of people and the number of times each one shuffles a deck, and how long playing cards have been around. It's better to give a number that is too big rather than too small.

3. Let's number the m shufflings in part (2) by $1, \ldots, m$. Let $A_{i,j}$ be the event that the ith shuffle and the jth shuffle give the same ordering of cards. What is $P(A_{i,j})$?

4. Let A be the event that two of the shufflings are the same, that is, $A = \bigcup_{i \neq j} A_{i,j}$ (the union of all the $A_{i,j}$'s for all pairs (i, j) with $i \neq j$). Show that $P(A) \le \sum_{i \neq j} P(A_{i,j}) = \binom{m}{2} P(A_{1,2})$.

5. Use a calculator to estimate $\binom{m}{2} P(A_{1,2})$.

6. How does the result from this problem contrast with the birthday problem? Why do you think this happens?