Math 180A Homework 3

Winter 2023

Due date: **11:59pm** (Pacific Time) on **Wed. Feb 1** (via Gradescope)

Section 1 (input directly in Gradescope)

Submit the answers to these problems directly through the Gradescope interface. You do not need to write up or explain your work.

Problem 1 (numerical answers)**.** Suppose that $P(A) = 0.3$ and $P(B) = 0.6$.

(a) If A and B are disjoint, then what is $P(A \cup B)$?

(b) If A and B are independent, then what is $P(A \cup B)$?

Problem 2 (numerical answer)**.** Suppose that A, B, and C are mutually independent, $P(A) = 0.2$, $P(B) = 0.3$, and $P(C) = 0.4$. Compute $P((A \cap B) \cup C)$. *Hint: Draw a Venn diagram or use inclusion-exclusion.*

Section 2 (upload files)

For each problem, write your solution on a page by itself, and upload it as a separate file to Gradescope (either typed or scanned from handwritten work). You should write your solutions to these problems neatly and carefully and provide full justification for your answers.

Problem 3. We choose a number uniformly at random from $1, \ldots, 10$. Call this number X. Let A be the event that X is divisible by 2, and let B be the event that X is divisible by 5. Are A and B independent? Explain.

Problem 4. In a school, there are four kindergarten classes, with 21, 24, 17, and 28 students respectively. We choose one of the 90 kindergarten students randomly. Let X denote the number of students in the class of the randomly selected student. One of the four kindergarten teachers is also randomly selected. Let Y denote the number of students in this teacher's class.

(a) Before explicitly calculating $E(X)$ and $E(Y)$, which do you think is larger? Why?

(b) Find $E(X)$ and $E(Y)$.

Problem 5. *(This is Exercise 3.10 from Anderson, Seppäläinen, and Valkó's book.)* Let X have probability mass function

$$P(X = -1) = \tfrac{1}{2},\ P(X = 0) = \tfrac{1}{3},\ \text{and } P(X = 1) = \tfrac{1}{6}$$

Calculate $E[|X|]$ using the two approaches in (a) and (b) below.

(a) First, find the probability mass function of the random variable $Y = |X|$, and using that, compute $E[|X|]$.

(b) Apply the formula given in class for the expectation of a function of a random variable:

$$\mathbb{E}(g(X)) = \sum_{k:\ p_X(k)>0} g(k) \cdot P(X = k)$$

Problem 6. We shuffle a deck of 52 cards and then turn them over one by one. Let X denote the number of times when we see two consecutive cards with the same rank (for example, two aces in a row or two eights in a row). Find the expected value of X.

Problem 7. *It's Saturday night, and you're in your friend Zach's mom's basement, playing Dungeons and Dragons. In the game, your character rounds a corner in a subterranean maze, only to find a hideous slime monster. You must roll a large number on your dice to defeat the monster!*

You roll two (independent) 20-sided dice. Let X be the maximum of the two numbers and let Y be the minimum of the two numbers on the dice. [1]

(a) *You have taken the slime monster by surprise! You can use the larger of your two dice rolls.* Find the probability mass function and cumulative distribution function of X.

Hint: starting with the cumulative distribution function and noticing that $P(X = k) = P(X \leq k) - P(X \leq k - 1)$ can save you some work.

(b) *The monster takes you by surprise! You must use the smaller of the two dice rolls.* Find the probability mass function of Y.

(c) (Bonus – NOT TO BE TURNED IN) Find $E[X]$ and $E[Y]$. *How much does the "element of surprise" help you?*

Problem 8 (Bonus – NOT TO BE TURNED IN)**.** Show that if a random variable X takes only nonnegative integers as its values, then

$$E(X) = \sum_{k=1}^{\infty} P(X \geq k).$$

Hint: write $P(X \geq k)$ as $\sum_{i=k}^{\infty} P(X = i)$ in the sum, and then switch the order of the two summations. (But be careful with the limits of the sums!)

[1] In fact, this is actually how the game Dungeons and Dragons works. In a variety of situations (e.g., fighting a monster, scaling a cliff, etc.), players roll 20-sided dice to determine how successful they will be, and they may have "advantage" or "disadvantage" requiring them to use the larger or smaller of two dice rolls.