Math 180A Homework 7

Winter 2023

Due date: **11:59pm** (Pacific Time) on **Wed., Mar. 1** (via Gradescope)

Section 1 (input directly in Gradescope)

Submit the answers to these problems directly through the Gradescope interface. You do not need to write up or explain your work.

Problem 1 (numerical answers)**.** The joint probability mass function of the random variables (X, Y) is given by the following table:

		Y			
		0	1	2	3
X	1	$\frac{1}{15}$	$\frac{1}{15}$	$\frac{2}{15}$	$\frac{1}{15}$
	2	$\frac{1}{10}$	$\frac{1}{10}$	$\frac{1}{5}$	$\frac{1}{10}$
	3	$\frac{1}{30}$	$\frac{1}{30}$	0	$\frac{1}{10}$

(a) Calculate the probability $P(Y - X \geq 1)$.

(b) Calculate the probability $P(X + Y^2 \leq 2)$.

Section 2 (upload files)

For each problem, write your solution on a page by itself, and upload it as a separate file to Gradescope (either typed or scanned from handwritten work). You should write your solutions to these problems neatly and carefully and provide full justification for your answers.

Problem 2. Suppose X, Y have joint density function

$$f(x, y) = \begin{cases} \frac{12}{7}(xy + y^2), & 0 \leq x \leq 1 \text{ and } 0 \leq y \leq 1 \\ 0, & \text{otherwise.} \end{cases}$$

(a) Find the marginal density functions of X and Y.

(b) Calculate the probability $P(X < Y)$.

(c) Calculate the expectation $E[X^2Y]$.

Problem 3. *(This is Exercise 6.29 from Anderson, Seppäläinen, and Valkó's book.)* Suppose that $X \sim \text{Geom}(p)$ and $Y \sim \text{Geom}(r)$ are independent. Find the probability $P(X < Y)$.

Problem 4. Suppose that X, Y are jointly continuous with joint probability density function

$$f(x,y) = \begin{cases} c(1-\cos x \cos y) & \text{if } \ x, y \in [0, \pi], \\ 0 & \text{else} \end{cases}$$

for some constant c.

(a) Find the value of the constant c.

(b) Find the marginal density functions of X and Y. Identify by name the (marginal) probability distributions of X and Y.

(c) Determine whether X and Y are independent.

Problem 5. You have a lamp with two different lightbulbs of different types. The lifetime of the first lightbulb (in years) is described by an exponential random variable X with mean 1, and the second lightbulb's lifetime is described by an exponential random variable Y with mean 2, independent from X. Let T be the amount of time from now until either one of the lightbulbs burns out, so $T = \min(X, Y)$.

(a) Find the cumulative distribution function of T.

(b) Find the probability density function of T.

(c) Find the expectation of T.

Problem 6. *(This is Exercise 7.2 from Anderson, Seppäläinen, and Valkó's book.)* Let X and Y be independent Bernoulli random variables with parameters p and r, respectively. Find the probability mass function of $X + Y$.

Problem 7. Let X have density $f_X(x) = 2x$ for $0 < x < 1$ and let Y be uniform on the interval $(1, 2)$. Assume X and Y are independent. Find the probability density function of $X + Y$.

Problem 8. (Bonus — 1 point) Prove that the sum of independent Poissons is Poisson: if X and Y are independent, with $X \sim \text{Pois}(\lambda)$ and $Y \sim \text{Pois}(\mu)$, then show that $X + Y \sim \text{Pois}(\lambda + \mu)$ by finding the PMF of $X + Y$. Is there a reason this makes sense intuitively? *(For a hint, see the next page.)*

Problem 9. (Bonus — 1 point) Prove that the sum of independent normals is normal: if X and Y are independent, with $X \sim \mathcal{N}(\mu_1, \sigma_1^2)$ and $Y \sim \mathcal{N}(\mu_2, \sigma_2^2)$, then show that $X + Y \sim \mathcal{N}(\mu_1 + \mu_2, \sigma_1^2 + \sigma_2^2)$ by finding the PDF of $X + Y$. *(For a hint, see the next page.)*

Hint for Problem 8: for one step, you may need to use the binomial theorem!

Hint for Problem 9: for one step, you may need to "complete the square."

Note: both problem Problem 8 and Problem 9 are graded for completion only, so feel free to double-check your work against online sources, but make sure you really solve them yourself and learn something!