1. (15) (a) State and prove the Schur Decomposition Theorem.

(b) Use it to prove: A has n orthonormal eigenvectors iff \( A^H A = A A^H \), where \( A \in \mathbb{C}^{n\times n} \).

2. (20) (a) Let \( A \) be \( m \times n, m > n, B = [A|z] \). Show that \( \sigma_1(B) \geq \sigma_1(A) \) and \( \sigma_{n+1}(B) \leq \sigma_n(A) \).

(b) Let \( A \) be \( m \times n, m \geq n, C = \begin{bmatrix} A \\ v^T \end{bmatrix} \). Show that \( \sigma_n(C) \geq \sigma_n(A) \) and \( \sigma_1(A) \leq \sigma_1(C) \leq \sqrt{\sigma_1(A)^2 + v^Tv} \).

3. (10) (a) Use Gershgorin's Theorem to prove that a real symmetric diagonally dominant matrix with positive diagonal elements is positive definite.

(b) Show that if the single shift QR method converges, then the convergence is:
   (a) quadratic for general matrices
   (b) cubic for symmetric matrices

4. (10) Prove that \( \| B(\lambda) - A^+ \|_2 = \frac{\lambda}{\sigma_r(\sigma_r^2 + \lambda)} \), where \( B(\lambda) = (A^T A + \lambda I)^{-1} A^T, \lambda > 0, A \) is \( m \times n, m \geq n, r = \text{rank}(A) \).

5. (10) Let \( A \) be \( n \times n \), nonsingular, and \( A = QR \), where \( Q \) is orthogonal and \( R \) is upper triangular with positive diagonal. Prove that \( Q \) and \( R \) are unique.

6. (15) Prove that if \( A \) is symmetric positive definite, \( \max_{i,j} |a_{ij}| = 1 \), then \( \max_{i,j,k} |a_{ij}^{(k)}| = 1 \) under \( LDL^T \) (or \( LU \)) decomposition.
Question 1. Let $f \in C^4(a, b)$, and let $x_0 = a < x_1 < \ldots < x_{n-1} < x_n = b$. Let $s$ be the $C^2$ natural cubic spline interpolant of $f$ and let $g$ be any other $C^2$ function satisfying $g(x_i) = f(x_i), 0 \leq i \leq n, g''(x_0) = g''(x_n) = 0$. Prove
\[ \| s'' \|_{L^2(a,b)} \leq \| g'' \|_{L^2(a,b)}. \]

Question 2. Find the one-point Gauss-Quadrature Rule of the form
\[ \int_0^1 f(x)\sqrt{x} \, dx \approx Af(\alpha). \]

Question 3. Define the terms:

a. Consistency
b. Stability
c. Convergence

as they relate to a multi-step formula for solving the initial value problem $y' = f(y), y(0) = y_0$. Apply these concepts to analyze the two step formula
\[ y_{k+1} = y_{k-1} + 2hf(y_k). \]