(1) Let a sequence of random variables be given by \( X_n = \frac{1}{n} \). Also, let \( f(x) = \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{if } x \leq 0 \end{cases} \). Show that \( X_n \overset{L}{\to} 0 \) but \( f(X_n) \not\overset{L}{\to} f(0) \).

(2) State a version of the Slutsky theorem under which it is true that \( X_n \overset{L}{\to} 0 \implies f(X_n) \overset{L}{\to} f(0) \).

In this connection, what goes wrong in Problem (1)?

(3) Suppose \( X_1, X_2, \ldots, X_n \) are i.i.d. random variables with sample mean and sample variance given by \( \overline{X}_n \) and \( S_n^2 \), respectively.

a. If the underlying distribution is \( N(\mu, \sigma^2) \), with both parameters unknown, what is the distribution of

\[
\frac{\sqrt{n-1} (\overline{X}_n - \mu)}{S_n}.
\]

Develop confidence intervals for \( \mu \) in this case.
b. Now suppose we have a random sample from any distribution with finite mean and variance. In this case, derive the limiting distribution of
\[ \frac{\sqrt{n-1}(\bar{X}_n - \mu)}{S_n} \].

Explain how the large sample confidence intervals for \( \mu \) differ from those obtained in part a.

c. What is the limiting distribution of
\[ \left( \frac{\sqrt{n-1}(\bar{X}_n - \mu)}{S_n} \right)^2 \]?

(Be certain to qualify your answer.)

d. Derive large sample confidence intervals for the variance \( \sigma^2 \).
Question (4) \( X \) is data from a statistical problem \( F \).

a. What does it mean for a statistic \( T(X) \) to be complete and sufficient for \( F \)?

b. State the factorization criterion for sufficiency.

c. Let \( X_1, \ldots, X_n (n \geq 3) \sim \text{IID } F \in \mathcal{F} \), the class of all densities on \( \mathbb{R} \). If \( \pi = P[X_1 \geq -1] \) find the UMVUE \( S \) of \( \pi^3 \).

d. (continuing c.) Consider the subclass \( \mathcal{F}_0 \subset \mathcal{F} \) consisting of uniform distributions \( \{U[\theta-1, \theta+1]\}_{\theta \in \mathbb{R}} \). If it is known that \( F \in \mathcal{F}_0 \), explain whether \( S \) above is still UMVUE for this problem.

c. Question (5). \( [x_1] \) and \( [y_2] \) are independent.

\[ \text{cov}(x_1, x_2) = \text{cov}(y_1, y_2) = \rho. \] Show that \( \text{cov}(x_1 + x_2; y_1 + y_2) = \rho. \)
Could the same conclusion hold if, instead of independence, all four random variables were positively correlated?

Question (6). Consider estimating \( \xi^2 \) from iid. data

\( X_i \sim N(\xi, 1); \ i = 1, \ldots, n \). Estimator \( T_1 = \bar{X}^2 \)

a. What does Jensen's inequality tell you about the bias of \( T_1 \)?

b. Explain whether \( T_1 \) is the MLE for \( \xi^2 \)?
c. In what sense is the bias of $T_1$ "removable"?

d. Construct an unbiased estimator $T_2$ as an observable function of $T_1$ and explain whether $T_2$ is

(i) UMVU; (ii) admissible or not.

e. Show that if $\xi \neq 0$, $T_1$ and $T_2$ are asymptotically equivalent in the sense of MSE risk. (If you wish assume the formula $\text{var } X^2 = \frac{2}{n^2} + 4\frac{\xi}{n}$, although there is small extra credit for deriving it.)