Numerical Analysis Qualifying Examination

September 8, 2006

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<td>1. Show how to construct the Lagrange interpolant $p_n(x)$ satisfying $p_n(x_i) = f(x_i)$, $0 \leq i \leq n$ using divided differences.</td>
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<td>2. Prove $f(x) - p_n(x) = f[x, x_0, x_1, \ldots, x_n] \prod_{i=0}^{n} (x - x_i)$.</td>
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<td>3. a. Compute the knots and weights such that $Q(f)$ is the two point Gaussian quadrature formula.</td>
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<td>b. Determine the order of the quadrature formula computed in part a.</td>
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<td>c. Write down an expression for the error $</td>
<td>I(f) - Q(f)</td>
<td>$.</td>
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Question 1. In this problem we will analyze the case of Lagrange interpolation on a set of distinct knots $x_0 < x_1 < \ldots < x_n$ with corresponding function values $f(x_i)$, $0 \leq i \leq n$.

a. Show how to construct the Lagrange interpolant $p_n(x)$ satisfying $p_n(x_i) = f(x_i)$, $0 \leq i \leq n$ using divided differences.

b. Prove $f(x) - p_n(x) = f[x, x_0, x_1, \ldots, x_n] \prod_{i=0}^{n} (x - x_i)$.

Question 2. Define the terms:

a. Consistency
b. Stability
c. Convergence

as they relate to a multistep formula. Apply these concepts to analyze the two step formula $y_{k+1} = y_{k-1} + 2hf(y_k)$.

Question 3. Let $I(f) = \int_{-1}^{1} f(x) \approx w_1 f(x_1) + w_2 f(x_2) = Q(f)$

a. Compute the knots and weights such that $Q(f)$ is the two point Gaussian quadrature formula.

b. Determine the order of the quadrature formula computed in part a.

c. Write down an expression for the error $|I(f) - Q(f)|$. 
1. Let the computed $L$ and $U$ satisfy $A + E = LU$, where $L$ is unit lower triangular and $U$ is upper triangular. Derive the bound on $E$: $|E_{ij}| \leq (3 + u)u \max(i - 1, j)g$, $g = \max \max |a_{ij}^{(k)}|$.

2. Prove that $\hat{x}$ is a least squares solution to $r = Ax - b$, where $A$ is $m \times n$ and $m \geq n$, iff $\hat{x}$ satisfies the normal equations.

3. (a) Prove that if $A$ is positive semi-definite, then its eigenvalues are non-negative.
(b) Prove that if $A$ is real symmetric, then $A$ is positive definite iff its eigenvalues are positive.
(c) Let $B = \begin{bmatrix} A \\ a^T \end{bmatrix}$, where $A$ is $m \times n$, $m \geq n$. Prove that $\sigma_n(B) \geq \sigma_n(A)$ and $\sigma_1(A) \leq \sigma_1(B) \leq \sqrt{\sigma_1(A)^2 + \|a\|^2}$.

4. Let $A$ be $m \times n$.
(a) Prove that if $A^+ = \begin{cases} (A^T A)^{-1}A^T & \text{if } \text{rank}(A) = n \\ A^T (AA^T)^{-1} & \text{if } \text{rank}(A) = m. \end{cases}$
(b) Prove that $\|B(\lambda - A^+\|_2 = \frac{\lambda}{\sigma_r(\sigma^2 + \lambda)}$, where $B(\lambda) = (A^TA + \lambda I)^{-1}A^T$, $\lambda > 0$, $m \geq n$, rank$(A) = r$.

5. Let $A$ be symmetric positive definite.
(a) Prove that $a_{ii} > 0$ for all $i$ and $|a_{ij}| < (a_{ii} + a_{jj})/2$ for $i \neq j$.
(b) Prove that $A = LDL^T$ exists, where $L$ is unit lower triangular and $D$ is diagonal with positive diagonal elements.
(c) Prove that $\max \max |a_{ij}^{(k)}| = \max |a_{ij}|$.