"[n]" means the problem is worth n points.

1. Let $G$ be a group of order $240 = 2^4 \cdot 3 \cdot 5$.
   a [10]. How many $p$-Sylow subgroups might $G$ have, for $p = 2, 3, 5$?
If $G$ has a subgroup of order 15, show that it has an element of order 15.
c [15]. Say $G$ doesn't have a subgroup of order 15.
Show that the number of 2-Sylows is 10 or 40.
2. Let $R$ denote a commutative ring and $I$ an ideal, $I \neq R$. We say that $R$ has nilpotents if $\exists r \in R, n \in \mathbb{N}, r \neq 0, r^n = 0$.

a [10]. Give an example where $R/I$ has nilpotents but $R$ doesn't.
b [10]. Give an example where $R$ has nilpotents but $R/I$ doesn't.
3. Let $\phi : \mathbb{C}[x] \rightarrow F$ be a ring homomorphism, where $F$ is a field, $\phi(1) \neq 0$. Give an example where $\phi$ is not onto.
b [20]. If \( \phi \) is core, show that \( F \cong \mathbb{C} \).
4a [10]. Give an example of two finitely generated $\mathbb{Z}$-modules, $M$ and $N$, such that $M, N$ are not isomorphic (as $\mathbb{Z}$-modules) but $\mathbb{Q} \otimes_{\mathbb{Z}} M \cong \mathbb{Q} \otimes_{\mathbb{Z}} N$ (as $\mathbb{Q}$-modules).
4b [10]. Let $M$ be a finitely generated $R[x]$-module, described using the classification of f.g. modules over a PID. Give a similar description of $C[x] \otimes_R M$ as a $C[x]$-module.
4c [10]. Show that if $M, N$ are two finitely generated $\mathbb{R}[x]$-modules, and $\mathbb{C}[x] \otimes_{\mathbb{R}[x]} M \cong \mathbb{C}[x] \otimes_{\mathbb{R}[x]} N$ (as $\mathbb{C}[x]$-modules), then $M \cong N$ (as $\mathbb{R}[x]$-modules).
5 [15]. Let $F$ be a field of characteristic $p$, and $f \in F[x]$ a polynomial. $f(x) = \sum_j f_j x^j$. Give necessary and sufficient conditions on the \{f_i\} for $f(x^p)$ itself to be a $p$th power, i.e. $\exists g(x)$ s.t. $f(x^p) = g(x)^p$. In particular, prove that your condition is necessary.
6. Let $F \supseteq K$ be an extension field of degree 2.
If $K$ is characteristic not 2, show $F$ is Galois over $K$. 
b [5]. Give an example where $F$ is Galois over $K$ even though $\text{char } K = 2$. 
c [10]. Give an example where \( F \) is not Galois over \( K \).