1. (a) (10) Prove that if $T \in \mathbb{C}^{n \times n}$,

$$T = \begin{bmatrix} T_{11} & T_{12} \\ 0 & T_{22} \end{bmatrix}_p,$$

then $\lambda(T) = \lambda(T_{11}) \cup \lambda(T_{22})$.

(b) (20) Prove that if $A \in \mathbb{C}^{n \times n}$, $B \in \mathbb{C}^{p \times p}$, and $X \in \mathbb{C}^{n \times p}$ satisfy $AX = XB$, rank($X$) = $p$, then there exists a unitary $Q \in \mathbb{C}^{n \times n}$ such that

$$Q^H A Q = T = \begin{bmatrix} T_{11} & T_{12} \\ 0 & T_{22} \end{bmatrix}_p.$$

where $\lambda(T_{11}) = \lambda(A) \cup \lambda(B)$. (Hint: consider QR decomposition of $X$, and use (a).)

2. (a) (25) State and prove the Schur Decomposition Theorem for $A \in \mathbb{C}^{n \times n}$. (Hint: use induction and 1(b).)

(b) (10) Use 2(a) to prove that $A \in \mathbb{C}^{n \times n}$ has $n$ orthonormal eigenvectors iff $A^H A = AA^H$.

3. (a) (25) State and prove the SVD Existence Theorem for $A \in \mathbb{R}^{m \times n}$.

(b) (15) Let $A \in \mathbb{R}^{m \times n}$, $A = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}$, where $A_{11}$ is $k \times k$. Use $S = \begin{bmatrix} A_{11} & A_{12} \\ 0 & 0 \end{bmatrix}$.

$m \times n$, to show that $\sigma_{k+1}(A) \leq \|A_{22}\|_2$.

(c) (10) Let $A \in \mathbb{R}^{m \times n}$, $m \geq n$. Show that there exist an orthogonal $Q$ and a symmetric positive semi-definite $P$ such that $A = QP$.

4. (a) (20) Prove that $\hat{x}$ is a least squares solution to $r = Ax - b$ iff $\hat{x}$ satisfies the normal equations, where $A$ is $m \times n$, $m \geq n$.

(b) (15) Let $A$ by $n \times n$, symmetric positive definite. Let $u_1, \ldots, u_n$ be an orthonormal basis of eigenvectors corresponding to $\lambda_1, \ldots, \lambda_n$. Let $w = \sum_{j=1}^n \sigma_j u_j$, $S(\mu) \equiv (A - \mu I)^{-1} w$, $\mu > 0$. Show that

$$\frac{d}{d\mu} \|S(\mu)\|_2 = -\frac{S(\mu)^T (A + \mu I)^{-1} S(\mu)}{\|S(\mu)\|_2}. $$
Part C: Approximation, Interpolation, and Numerical Quadrature. We assume that $a, b \in \mathbb{R}$ with $a < b$. For any integer $n \geq 0$, we denote by $\mathcal{P}_n$ the set of all polynomials of degree $\leq n$ and by $\mathcal{P}_n^1$ the set of all polynomials in $\mathcal{P}_n$ with leading coefficient 1.

Question 3.1 [20 points]

Let $n \geq 1$ be an integer. Let $Q_k \in \mathcal{P}_n^k$ ($k = 0, \ldots, n$) be such that

$$\int_a^b Q_j(x)Q_k(x)\,dx = 0$$

for any indices $j$ and $k$ with $0 \leq j < k \leq n$. Let $P_n \in \mathcal{P}_n$. Prove the following:

(a) The identity

$$P_n(x) = c_0 Q_0(x) + \cdots + c_{n-1} Q_{n-1}(x) + Q_n(x)$$

holds true for a unique set of real numbers $c_0, c_1, \ldots, c_{n-1}$. Moreover,

$$\int_a^b |P_n(x)|^2\,dx = c_0^2 \int_a^b |Q_0(x)|^2\,dx + \cdots + c_{n-1}^2 \int_a^b |Q_{n-1}(x)|^2\,dx + \int_a^b |Q_n(x)|^2\,dx.$$

(b) The inequality

$$\int_a^b |Q_n(x)|^2\,dx \leq \int_a^b |P_n(x)|^2\,dx$$

holds true. Moreover, this inequality becomes equality if and only if $P_n = Q_n$.

Question 3.2 [20 points]

(1) Calculate the Lagrange interpolation polynomial that interpolates the function $f(x) = x^{10}$ at points $x = 0, 1, \ldots, 20$. Justify your answer.

(2) Let $N \geq 1$ be an integer, $h = (b - a)/N$, and $x_j = a + jh$ ($j = 0, \ldots, N$). The composite mid-point quadrature is given by

$$\int_a^b f(x)\,dx \approx h \sum_{j=1}^N f\left(\frac{x_{j-1} + x_j}{2}\right).$$

Suppose $f \in C^2[a, b]$. Prove that there exists $\xi \in [a, b]$ such that

$$\int_a^b f(x)\,dx - h \sum_{j=1}^N f\left(\frac{x_{j-1} + x_j}{2}\right) = \frac{1}{24} (b - a) h^2 f''(\xi).$$