ALGEBRA QUALIFYING EXAM
September, 2010

Do 8 Problems

(1) Show that a finitely generated subgroup of the additive group of the rationals is cyclic.

(2) Show that a group of order $2010 = 2 \times 3 \times 5 \times 67$ is solvable.

(3) Show that if $H$ is a cyclic normal subgroup of $G$, then every subgroup of $H$ is normal in $G$.

(4) Let $E$ be a finite separable extension of $F$. Show that, then $E = F(a)$, for some $a$ in $E$. (Hint: Use the Fundamental Theorem of Galois Theory)

(5) Let $E$ be a finite dimensional Galois extension of a field $F$ and let $G = Gal(E/F)$. Suppose that $G$ is an abelian group. Prove that if $K$ is any field between $E$ and $F$, then $K$ is a Galois extension of $F$. What is the Galois group of $K$ over $F$?

(6) Explicitly determine the splitting fields over the rationals of the following two polynomials and their degrees over $Q$:
   (a) $x^6 + 1$ and
   (b) $x^6 - 1$

(7) Let $R$ be a commutative ring with identity and let $U$ be maximal among non-finitely generated ideals of $R$. Prove $U$ is a prime ideal.

(8) Let $R$ be a ring with identity such that the identity map is the only ring automorphism of $R$. Prove that the set $N$ of all nilpotent elements of $R$ is an ideal of $R$. (Hint: $1 + n$, with $n$ a nilpotent element, is invertible.)

(9) Give an example of a right noetherian ring that is not left noetherian and an example of a module that satisfies the descending chain condition on submodules, but not the ascending chain condition on submodules.