Problem 1 Let $X_1, ..., X_n$ be i.i.d. with density function

$$f(x; \theta) = \theta x^{\theta - 1}, \quad 0 < x < 1, \theta > 0.$$ 

This is a sub-family of the Beta distribution. Let $T = -\sum_{i=1}^n \log(X_i)/n$.

(a) Verify that this is an exponential family, write down the natural parameter and the sufficient statistic; (5 points)

(b) Use properties of the exponential family to show that $E(T) = 1/\theta$ and $\text{Var}(T) = 1/n\theta^2$; (10 points)

(c) Compute the Fisher information for this problem; (10 points)

(d) Deduce that $T$ achieves the information inequality lower bound, and therefore is UMVU for $1/\theta$. (5 points)

Problem 2 Let $X_1, ..., X_n$ be i.i.d. with density function

$$f(x; \theta) = \theta x^{\theta - 1} \exp(-x^\theta), \quad x > 0, \theta > 0.$$ 

This is a sub-family of the Weibull distribution.

(a) Explain if it belongs to the location, scale, or exponential family. (5 points)

(b) Show that there is a unique interior maximum of the likelihood function. (5 points)

(c) Find the maximum likelihood estimate $\hat{\theta}$ given the data. (5 points)

(d) Estimate the variance of $\hat{\theta}$. (5 points)

Problem 3 Suppose $X_1, ..., X_n$ is an i.i.d. sample from $\mathcal{N}(0, \sigma^2)$. We are interested in testing $H : \sigma \leq \sigma_0$ versus $K : \sigma > \sigma_0$.

(a) Fix $\sigma_1 > \sigma_0$. Write down the likelihood ratio for $H$ versus $K_1 : \sigma = \sigma_1$ and deduce that the test that rejects for large values of $\sum_i (X_i - \xi_0)^2$ is most powerful. (10 points)

(b) Give an explicit form for a UMP test for $H$ versus $K$ at level $\alpha \in (0,1)$. (10 points)

(c) Is this the only UMP test? (5 points)

Problem 4 Suppose $X_1, ..., X_n$ are independent with $X_i$ having the Poisson distribution with mean $\lambda_i$. Consider testing $H : \sum_i \lambda_i \leq a$ versus $K : \sum_i \lambda_i > a$, where $a > 0$ is fixed.

(a) Fix an alternative $(\lambda'_1, ..., \lambda'_n)$ with $\sum_i \lambda'_i > a$. Write down the likelihood ratio. (5 points)
(b) For a given alternative ($\lambda'_1, \ldots, \lambda'_s$), consider the prior on ($\lambda_1, \ldots, \lambda_s$) where $\lambda_i = a\lambda'_i / \sum_j \lambda'_j$. Give a most power test for this simple versus simple situation. (5 points)

(c) Show that the power of the test that rejects for large values of $\sum_i X_i$ is monotone in $\sum_i \lambda_i$. (5 points)

(d) Deduce a UMP test for $H$ versus $K$. (5 points)
Problem 5 \( X_1, \ldots, X_n \) iid uniform \([x-1, x+1]\) \((x > 1)\).

Do an ARE comparison of \( \hat{X}_1 = \bar{X} \) and \( \hat{X}_2 = \text{med} \frac{1}{3} X_i \frac{1}{3} \).

Are both limit-laws and risk comparisons available as a basis for this ARE determination? If so pick either one for your analysis. What size \( n \) does an \( \hat{X}_2 \)-statistician need to match the performance of an \( \hat{X}_1 \)-statistician using \( n = 1 \) million observations? Note with comment that neither estimator is optimal. What would happen if you compared \( \hat{X}_3 = \hat{X}_{\text{MLE}} \) and (bonus points) \( \hat{X}_4 = e^Y \) (where \( Y = \ln X_i \)) against \( \hat{X}_2 \) in an asymptotic analysis?

Prove that at least one of these estimators is inadmissible.