Problem 1. (10 pts)

Consider a simple group $G$ with 60 elements. Show that if $G$ has a subgroup $H$ of order 12 then $G \cong A_5$.

(Any simple group of order 60 is isomorphic to $A_5$, but obviously you cannot use this fact, unless you prove it.)

Problem 2. (10 pts)

Let $n \geq 3$ be an integer. Calculate the number of ordered pairs of permutations $(\sigma, \tau)$ in the symmetric group $S_n$ such that $\sigma \tau = \tau \sigma$. Your answer should be a simple formula involving known functions of $n$.

Problem 3. (10 pts)

Let $A$ be a commutative ring with unity, and assume that the elements $f_1, \ldots, f_n \in A$ generate the unit ideal (1). Show that there exists an injective ring homomorphism

$$\phi : A \to \prod_{i=1}^{n} A_{f_i}.$$ 

As usual, $A_f$ denotes the localization of $A$ at the set of powers of $f$.

Problem 4. (10 pts)

Assume that $A$ is a commutative Noetherian ring, and let $a_n \in A$ for $n \geq 0$. Prove that the power series

$$f = \sum_{n=0}^{\infty} a_n x^n$$

is nilpotent if and only if each $a_n \in A$ is nilpotent.

Problem 5. (10 pts)

Let $F$ be a field and $f(x) = x^4 + bx^2 + c \in F[x]$, for some $b, c \in F$.

If $K$ is the splitting field of $f(x)$ over $F$, prove that the Galois group $\text{Gal}(K/F)$ is isomorphic to a subgroup of the dihedral group $D_4$ of order 8.
Problem 6. (15 pts)
(a). (10 pts) Suppose that $K \subseteq L \subseteq M$ are fields such that $L/K$ and $M/L$ are Galois, and that every automorphism of $L/K$ extends to an automorphism of $M$. Prove that $M/K$ is Galois.

(b). (5 pts) Give an example of fields $K \subseteq L \subseteq M$ such that $L/K$ and $M/L$ are Galois, but $M/K$ is not Galois.

Problem 7. (15 pts)
(a). (10 pts) Let $M$ be a (left) module over a commutative ring $R$ and let $I$ be an ideal of $R$. Prove that
$$\left(\frac{R}{I}\right) \otimes_R M \cong M/IM$$
(as $R$-modules).

(b). (5 pts) Now let $R$ be a PID and let $I$ be a maximal ideal of $R$. Suppose that $M$ is a finitely generated $R$-module. Calculate the dimension of $\left(\frac{R}{I}\right) \otimes_R M$ as a vector space over the field $K = R/I$ in terms of the elementary divisors (or invariant factors) of $M$.

Problem 8. (10 pts)
Find the Jordan canonical form for the matrix
$$\begin{pmatrix}
0 & 1 & 0 \\
0 & 0 & 1 \\
1 & 0 & 0
\end{pmatrix}$$

(a). (5 pts) Over the complex numbers.

(b). (5 pts) Over the algebraic closure of the field of three elements.