Ph.D./Masters Qualifying Examination in Numerical Analysis

Examiners: Philip E. Gill and Michael Holst

9:00am–12:00pm
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5829 AP&M

| #1.1 | 20 |
| #1.2 | 20 |
| #1.3 | 20 |
| #2.1 | 20 |
| #2.2 | 20 |
| #2.3 | 20 |
| #3.1 | 20 |
| #3.2 | 20 |
| **Total** | **160** |

- Add your name in the box provided and staple this page to your solutions.
- Write your name clearly on every sheet submitted.
1. Norms, Condition numbers and Linear Equations

Question 1.1.

(a) Let $\Delta = \text{diag}(\delta_1, \delta_2, \ldots, \delta_n)$. Prove that for all $1 \leq p \leq \infty$,

$$\|\Delta\|_p = \max_{1 \leq i \leq n} |\delta_i|.$$  

(b) Let $A$ and $B$ be any pair of matrices such that the product $AB$ is defined. Prove that $\|AB\|_F \leq \|A\|_2 \|B\|_F$.

(c) Let $\| \cdot \|$ and $\| \cdot \|_D$ denote any vector norm and its corresponding dual norm. If $A \in \mathbb{C}^{n \times n}$, let $\|A\|_D$ denote the matrix norm subordinate to $\| \cdot \|_D$. Prove that if $x, y \in \mathbb{C}^n$ then

$$\|xy^H\| = \|x\| \|y\|_D.$$  

Question 1.2. Assume $A \in \mathbb{R}^{n \times n}$ is nonsingular. Find a solution $E^*$ of the problem

$$\min_{E \in \mathbb{R}^{n \times n}} \left\{ \frac{\|E\|_2^2}{\|A\|_2} \mid A + E \text{ singular} \right\}.$$ 

Give the optimal value of $\|E\|_2 / \|A\|_2$. Comment on the uniqueness of $E$.

Question 1.3. Let $A \in \mathbb{R}^{n \times n}$ be symmetric positive definite.

(a) Prove that if Gaussian elimination without interchanges is applied to $A$, then the remaining matrix is symmetric positive definite at every step. Hence prove that there exist $L$ and $U$ such that $A = LU$.

(b) If $A$ is factorized using Gaussian elimination without interchanges, prove that the growth bound satisfies $\rho_n \leq 1$. 
2. Least-Squares and Eigenvalues

Question 2.1. Assume that $A \in \mathbb{R}^{m \times n}$.

(a) Prove that every $y \in \mathbb{R}^m$ has a unique decomposition $y = y_R + y_N$, with $y_R \in \text{range}(A)$ and $y_N \in \text{null}(A^T)$.

(b) For any $x$, let $r$ denote the residual vector $b - Ax$. Prove that $x$ solves the least-squares problem $\min \|b - Ax\|_2$ if and only if $Ax = b_R$ and $r = b_N$.

(c) Assume that $A$ has full column rank.

(i) Prove that the least-squares solution is unique.

(ii) Describe how you would compute the least-squares solution using the QR decomposition of $A$.

Question 2.2. Consider a non-defective matrix $A \in \mathbb{C}^{2 \times 2}$ such that

$$A = \begin{pmatrix} a & c \\ 0 & b \end{pmatrix}.$$ 

(a) Find the left and right eigenvectors of $A$.

(b) Find the condition number of each of the eigenvalues of $A$.

Question 2.3. Let $A \in \mathbb{C}^{m \times n}$. Given an approximate eigenpair $(\lambda, u)$, describe how you would use one step of inverse iteration to find an improved eigenvector $v$ of $A$. Hence show that $(\lambda, v)$ is an exact eigenpair of $A + E$ where $E$ may be chosen to satisfy

$$\|E\|_F = \frac{\|u\|_2}{\|v\|_2}.$$
3. Interpolation, Approximation and ODEs

Question 3.1. Let \( f(x) \in C^\infty([a, b]) \), let \( p_n(x) \) interpolate \( f(x) \) at the \( n + 1 \) points \( a = x_0 < x_1 < \cdots < x_n = b \), and denote \( \mathcal{I}(f) = \int_a^b f(x)dx \).

(a) Show that the error in the interpolating polynomial can be written as:

\[
e_n(x) = f(x) - p_n(x) = f[x_0, x_1, \ldots, x_n, x] \psi_n(x),
\]

where \( \psi_n(x) = \prod_{i=0}^n (x - x_i) \), and where \( f[x_0, x_1, \ldots, x_n, x] \) is the divided difference at the \( n + 2 \) points \( x_0, x_1, \ldots, x_n, x \).

(b) Prove that

\[
f[x_0, x_1, \ldots, x_n, x] = \frac{f^{(n+1)}(\xi)}{(n + 1)!}, \quad \xi \in (a, b).
\]

(c) If \( \int_a^b \psi_n(x)dx = 0 \), show that

\[
\mathcal{I}(e_n) = \mathcal{I}(f) - \mathcal{I}(p_n) = \frac{f^{(n+2)}(\eta)}{(n + 2)!} \int_a^b \psi_n(x)dx, \quad \eta \in (a, b).
\]

Question 3.2. Consider the problem of best \( L^2 \)-approximation of \( u \in C^\infty([a, b]) \) from a subspace \( V \): Find \( u^* \in V \) such that

\[
\|u - u^*\|_{L^2([a,b])} = \inf_{v \in V} \|u - v\|_{L^2([a,b])}.
\]

(a) Derive a bound on the error in the best \( L^2 \)-approximation of the form:

\[
\|u - u^*\|_{L^2([a,b])} \leq C h^3 \|u^{(3)}\|_{L^2([a,b])}, \quad h = b - a,
\]

where \( u^* \) is chosen from the subspace of quadratic functions \( V \).

(b) Given now the particular function \( u(x) = x^3 \) over the particular interval \([a, b] = [0, 1]\), determine the best \( L^2 \)-approximation from the subspace of quadratic functions, and justify the technique you use.