May 2004

QUALIFYING EXAM
ALGEBRA
Parts II and III

Part II

1. (10 points) Prove the 3rd Sylow Theorem. Suppose $p$ is a prime dividing the order of a group $G$. Then the number of $p$-Sylow subgroups divides the order of $G$ and is congruent to $1 \mod p$. You may use the first two Sylow Theorems without proof.

2. (20 points) Let $G$ be the group of $2 \times 2$ invertible matrices with entries in the finite field $\mathbb{Z}_p$. Then we know that $|G| = (p - 1)^2 p(p + 1)$. Assume that $p = 17$. Then $|G| = 2^9 3^2 17$.
   a. Let $x$ be an element of $G$ of order 17. Prove that $x$ is conjugate to an element of the form $\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$.
   b. Prove that $G$ contains 18 Sylow 17-subgroups. Hint: Use the fact that the upper triangular matrices contain a Sylow 17-subgroup as a normal subgroup.
   c. How many elements in $G$ have order 17?

3. (10 points) Construct a nonabelian group of order $75 = 5^2 \cdot 3$.

Part III

4. (20 points) Let $n_i, 0 \leq i \leq m$ be integers. Use the Chinese Remainder Theorem to prove there exists a unique polynomial $f(X) \in \mathbb{Q}[X]$ of degree $\leq m$ with $f(i) = n_i, 0 \leq i \leq m$.

5. (10 points) In a commutative ring with 1 prove that every ideal is contained in a maximal (proper) ideal.

6. (20 points) Prove that a projective $R$-module is flat. Hint: First prove the case for a free $R$ module.

7. (20 points)
   a. Determine the Galois group $G$ for the splitting field $K$ of $X^5 - 3$ over $\mathbb{Q}$.
   b. Determine all the subgroups of $G$ isomorphic to $\mathbb{Z}_6$.

8. (20 points) Let $\Phi = X^2 + X + 1$ be the third cyclotomic polynomial.
   a. Prove that $\Phi$ is reducible over $\mathbb{Z}_7$ and give a complete factorization.
   b. Determine the possible rational canonical forms for any element $A \in GL(2, \mathbb{Z}_7)$ which satisfies $A^3 = 1$.

9. (10 points) Let $V$ and $W$ be finite dimensional vector spaces over the field of complex numbers of dimensions $m$ and $n$. Use the universal mapping property to prove that $V \otimes W$ is a vector space of dimension $mn$.

10. (10 points) In the Gaussian integers the norm is used to analyze sums of squares. Use this technique to determine how many ways $N = 3^4$ and $M = 5^4$ can each be expressed as the sum of two integer squares.