(15) 1. State and prove the Schur Decomposition Theorem.

(13) 2. (a) Let \( A \in M_n \). Show that if \( z^H A z = 0 \) for all \( z \in \mathbb{C}^n \) then \( A = 0 \).
    (b) Give a \( 2 \times 2 \) example of \( A \in M_2(\mathbb{R}) \), where \( A \neq 0 \), but \( x^T A x = 0 \) for all \( x \in \mathbb{R}^2 \).
    (c) Show that \( A \in M_n \) is normal iff \( \| Ax \|_2 = \| A^H x \|_2 \) for all \( z \in \mathbb{C}^n \).
    (d) Show that \( A \in M_n \) is normal iff \( \theta(Ax, Ay) = \theta(A^H x, A^H y) \) for all \( x, y \in \mathbb{C}^n \).

(12) 3. Let \( \tilde{x} \) be a least squares solution to \( Ax = b \), where \( A \in M_{m,n} \) and \( m \geq n \). Let \( A^\dagger \) be the pseudo-inverse of \( A \). Use the Singular Value Decomposition to show that \( \tilde{x} = A^\dagger b \) is the min 2-norm least squares solution to \( Ax = b \), i.e., show
    (a) \( \tilde{x} \) is a least squares solution.
    (b) if \( \hat{x} \) is a least square solution then \( \| \hat{x} \|_2 \geq \| \tilde{x} \|_2 \), and
    (c) \( \tilde{x} \) is unique.
Let $N = \{0, 1, 2, \ldots \}$, $Z = \{0, \pm 1, \pm 2, \ldots \}$, $Q$ equal the rationals and $C$ denote the complex numbers.

If $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k)$ is a partition of $n$, let $A^\lambda$ denote the irreducible representation of the symmetric group $S_n$ such that the Frobenius image of $x^\lambda$ is the Schur function $S_{\lambda(x_1, \ldots , x_N)}$ where $N > n$.

(1) (30 pts) Let $G$ and $H$ be a finite groups, $A : G \to GL(n, \mathbb{C})$ be a representation of $G$ and $B : H \to GL(m, \mathbb{C})$ be a representation of $H$.

(a) Define the representation $A \times B : G \times H \to GL(n \cdot m, \mathbb{C})$
(b) Show that if $A$ and $B$ are irreducible representations, then $A \times B$ is irreducible.
(c) Show that every irreducible representation of $G \times H$ is of the form $A \times B$ where $A$ is an irreducible representation of $G$ and $B$ is an irreducible representation of $H$.

(2) (30 pts)
(a) Let $T$ be the trivial representation. Decompose $T \uparrow_{S_3 \times S_3}^{S_6}$ as a sum of irreducible representations of $S_6$ where $S_3 \times S_3$ is the Young subgroup of $S_6$ consisting of all permutations $\sigma \in S_6$ such that
\[ \sigma(1), \sigma(2), \sigma(3) \in \{1, 2, 3\}, \sigma(4), \sigma(5), \sigma(6) \in \{4, 5, 6\}. \]
(b) Decompose $A^{(2,4)} \otimes A^{(1,5)}$ as a sum of irreducible representations of $S_6$ where $\otimes$ represents the Kronecker product of the representations.
(c) Decompose $A^{(1,2)} \times A^{(1,2)} \uparrow_{S_3 \times S_3}^{S_6}$ as a sum of irreducible representations of $S_6$. (Here $S_3 \times S_3$ is group described in part (a).)

(3) (30 pts)
(a) Use the Murnaghnam-Nakayama rule to compute the values of the character $A^{(1,4)}$ on the conjugacy classes of $S_5$.
(b) Express $\chi^{A^{(1,4)}}_{S_3 \times S_2}$ as a sum of irreducible characters of $S_3 \times S_2$. Here $S_3 \times S_2$ is the Young subgroup of $S_6$ consisting of all permutations $\sigma \in S_6$ such that
\[ \sigma(1), \sigma(2), \sigma(3) \in \{1, 2, 3\}, \sigma(4), \sigma(5) \in \{4, 5\}. \]

(4) (30 pts) Let $G$ be the group of order 8 defined by the relations
\[ a^4 = 1 \text{ and } a^2 = b^2 = 1 \text{ and } b^{-1}ab = a^3. \]
(a) Show that $ab = b^2a$ and that every element of $G$ is of the form $b^k$ or $b^ka$ where $k = 0, \ldots , 3$.
(b) Given that the conjugacy classes of $G$ are
\begin{align*}
C_1 &= \{1\} \\
C_2 &= \{b^2\} \\
C_3 &= \{b, b^3\} \\
C_4 &= \{a, b^2a\} \\
C_5 &= \{ba, b^3a\}
\end{align*}
(c) Show that $H = \{1, b^2\}$ is a normal subgroup of $G$ for which $G/H$ is isomorphic to $Z_2 \times Z_2$.
Give the character character table for the lifting of the 4 linear characters of $G/H$ to $G$.
(iii) Find the complete character table for $G$. 

1
Let $H$ be a subgroup of $G$ and let $G = r_1 H + \ldots + r_k H$ be its coset decomposition. Define a permutation representation $L$ of $G$ by

$$\sigma(r_1 H, \ldots, r_k H) = (\sigma r_1 H, \ldots, \sigma r_k H) = (r_1 H, \ldots, r_k H)L(\sigma)$$

so that

$$L(\sigma)_{ij} = \chi(\sigma r_i H = \sigma r_j H)$$

(a) Prove that $L$ is a representation.

(b) Consider the special case where $G = S_n$ and $H = S_{n-1} \times S_1 = \{ \sigma \in S_n : \sigma(n) = n \}$.

(i) Show that the coset decomposition of $G$ relative to $H$ is given by $G = H + (1, n) H + \ldots + (n-1, n) H$ where $(i, n)$ denotes the transposition which interchanges $i$ and $n$.

(ii) Show that $\chi^L(\sigma) = \fix(\sigma)$ where $\fix(\sigma)$ denotes the number of fixed points of $\sigma$.

(c) In the special case where $G = S_4$ and $H = S_3 \times S_1$, use part (b) to decompose $L$ a sum of irreducible representations of $S_4$.

(6) (30 pts) If $S = \{ 1 \leq i_1 < i_2 < \cdots < i_k \leq n \}$ is a subset of $\{1, 2, \ldots, n\}$ and $\sigma = (\sigma_1, \sigma_2, \ldots, \sigma_n)$ is a permutation let $\sigma(S)$ denote the subset $\sigma(S) = \{ \sigma_{i_1}, \sigma_{i_2}, \ldots, \sigma_{i_k} \}$. In this manner, we can define an action of $S_n$ on the $k$-subsets of $\{1, 2, \ldots, n\}$ and induce a representation $A^{(k,n)}$ such that if $S_1, S_2, \ldots, S_{(n)}$ is a list of the $k$-element subsets of $\{1, \ldots, n\}$, then

$$\sigma(S_1, \ldots, S_{(n)}) = (\sigma(S_1), \ldots, \sigma(S_{(n)})) A^{(k,n)}(\sigma).$$

Let $\chi^{(k,n)}$ the character of $A^{(k,n)}$. Find the Frobenius image of $\chi^{(2,4)}$.

(b) Use your result in (a) to compute the decomposition of $\chi^{(2,4)}$ into a sum of irreducible characters of $S_4$.

(7) (30 pts) Let $H$ be a subgroup of a finite group $G$ and $A : H \to GL(n, C)$ be a representation of $H$.

(a) Prove that for any character $\phi$ of $G$, $\langle \chi^{A \uparrow_H^G}, \phi \rangle_G = \langle \chi^A, \phi \downarrow_H^G \rangle$.

(b) Given an example to show that it is not always the case that if $A$ is irreducible, then $A \uparrow_H^G$ is irreducible.

(c) Prove that if $K$ is a subgroup of $H$ and $B : K \to GL(m, C)$ is a representation of $K$, then the representation $B \uparrow_K^H$ is similar to the representation $(B \uparrow_K^H) \phi_H^G$.

(8) (30 pts) Let $R = (R, +, \cdot)$ be an integral domain. Let $C$ be the additive group of $R$ generated by the indentity, that is, let $C$ be the smallest subgroup of $R$ such that $C$ contains $0$ and $1$ and $C$ is closed under $+$. Here $n \geq 0$, then we can define $n \cdot 1$ by induction as $0 \cdot 1 = 0$, $1 \cdot 1 = 1$ and $(n + 1) \cdot 1 = 1 + (n \cdot 1)$ and if $n < 0$, we define $n \cdot 1 = -((n) \cdot 1)$.

Show that $\phi : Z \to C$ defined by $\phi(n) = n \cdot 1$ is a surjective ring homomorphism.

(c) Prove that either $C$ is isomorphic to $Z$ or $C$ is isomorphic to $Z_p$ for some prime $p$. 

2
Consider the equations
\[ x^2 + 2y^2 = 2 \\
\]
\[ x^2 - xy + y^2 = 1 \]

(a) Let \( I \) be the ideal of \( \mathbb{C}[x,y] \) generated by these equations. Find the Groebner basis for \( I \) relative to lexicographic order where \( y > x \).

(b) Find a Groebner basis for \( \mathbb{C}[x] \cap I \).

(c) Find all solutions to these equations that lie \( \mathbb{C}^2 \).

(d) Find a vector space basis for \( \mathbb{C}[x,y]/I \).

(10) (40 pts) Let \( I \) and \( J \) be ideals in \( k[x_1, \ldots, x_n] \) where \( k \) is field.

(a) Show that \( \sqrt{I} = \sqrt{J} \).

(b) Show that \( \sqrt{I \cap J} = \sqrt{IJ} \).

(c) Is \( x^2 - y^2 \in \sqrt{(x^2 + x, x^2 - y)} \)?

(d) Is \( x^2 + y^2 \in \sqrt{(x + y, x^2 - y)} \)?

(11) (40 pts) Let
\[ A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \]

(a) Show that \( A \) generates a cyclic group \( G \) of order 4.

(b) Show that if \( f \in \mathbb{C}[x,y]^G \), then \( f \) can have no monomials of odd degree.

(b) Use Molien's Theorem to show that the Hilbert series of \( G \) is
\[ \phi_G(z) = \frac{1 + z^4}{(1 - z^2)(1 - z^4)}. \]

(c) Show that \( \mathbb{C}[x,y]^G \) is Cohen-Macauly by explicitly finding the generators and separators for \( \mathbb{C}[x,y]^G \).