Ph.D./Masters Qualifying Examination in Numerical Analysis

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9am–12 Noon
Wednesday May 25, 2005
5829 AP&M

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- Add your name in the box provided and staple this page to your solutions.
- Write your name clearly on every sheet submitted.
1. Norms, Condition numbers and Linear Equations

Question 1.1.

(a) Let $\Delta = \text{diag}(\delta_1, \delta_2, \ldots, \delta_n)$. Prove that for all $1 \leq p \leq \infty$,$$
\|\Delta\|_p = \max_{1 \leq i \leq n} |\delta_i|.
$$

(b) Let $A$ and $B$ be any pair of matrices such that the product $AB$ is defined. Prove that $\|AB\|_F \leq \|A\|_2 \|B\|_F$.

(c) Let $\| \cdot \|$ and $\| \cdot \|_D$ denote any vector norm and its corresponding dual norm. If $A \in \mathbb{C}^{n \times n}$, let $\| A \|_D$ denote the matrix norm subordinate to $\| \cdot \|_D$. Prove that if $x, y \in \mathbb{C}^n$ then$$
\|xy^H\| = \|x\| \|y\|_D.
$$

Question 1.2.

(a) Consider the subtraction $x = a - b$ of two real numbers $a$ and $b$ such that $a \neq b$. Suppose that $\tilde{a}$ and $\tilde{b}$ are the result of making a relative perturbation $\Delta a$ and $\Delta b$ to $a$ and $b$. Find the relative error in $\tilde{x} = \tilde{a} - \tilde{b}$ as an approximation to $x$ and hence find a condition number for the operation of subtraction. Assume that all calculations are done in exact arithmetic.

(b) State the standard rounding-error model for floating-point arithmetic. Given three representable numbers $a$, $b$ and $c$, compute the backward and forward relative error for the floating-point value $\hat{s}$ of the calculation $s = ab + c$. Describe a situation in which $\hat{s}$ has large forward error, but small backward error.

Question 1.3.

(a) Prove that every nonsingular symmetric matrix $A$ can be written in the form $PAP^T =LBL^T$, where $P$ is a permutation, $L$ is unit lower triangular and $B$ is a block-diagonal matrix with diagonal blocks of order at most one or two.

(b) Briefly describe the diagonal complete pivoting method for finding the factorization $PAP^T =LBL^T$. Show that $\|L\|$ is bounded independently of $A$. 

2. Least-Squares and Eigenvalues

Question 2.1. Let \( A \) be an \( m \times n \) with rank \( r \). Assume that \( b \in \text{range}(A) \).

(a) Derive necessary and sufficient conditions for \( x \) to be the least-length solution of \( Ax = b \) and prove that the least-length solution is unique.

(b) Define an algorithm for computing the general solution of \( Ax = b \) using the QR factorization of \( A^T \) with column interchanges.

(c) Use part (b) to define the least-length solution. Verify that your algorithm gives the solution of least length.

Question 2.2. Consider a non-defective matrix \( A \in \mathbb{C}^{2 \times 2} \) such that

\[
A = \begin{pmatrix} a & c \\ 0 & b \end{pmatrix}.
\]

(a) Find the left and right eigenvectors of \( A \).

(b) Find the condition number of each of the eigenvalues of \( A \).

(c) Briefly discuss the situation where \( A \) is close to a defective matrix.

Question 2.3. Let \( A \in \mathbb{C}^{n \times n} \). Given an approximate eigenpair \( (\lambda, u) \), describe how you would use one step of inverse iteration to find an improved eigenvector \( v \) of \( A \). Hence show that \( (\lambda, v) \) is an exact eigenpair of \( A + E \) where \( E \) may be chosen to satisfy

\[
\|E\|_F = \frac{\|u\|_2}{\|v\|_2}.
\]
3. Interpolation, Approximation and ODEs

**Question 3.1.** Consider the function \( f(x) = 2x^3 - x^2 + 1 \) on \([0, 2]\).

(a) Construct the (unique) quadratic interpolation polynomial \( p_2(x) \) which interpolates \( f(x) \) at \( x = 0, 1, 2 \).

(b) Derive a bound on the error \( |f(x) - p_2(x)| \) which is valid over the interval \([0, 2]\).

(c) Use Simpson’s rule based on \( p_2(x) \) to compute an approximation to

\[
I(f) = \int_0^2 f(x) \, dx,
\]

and give an expression for the error in the approximation.

(d) Derive a bound on the error in the finite difference approximation:

\[
f'(x) = \left[ \frac{f(x + h) - f(x - h)}{2h} \right].
\]

**Question 3.2.** Consider the problem of best \( L^p \)-approximation of a (continuous) function \( u(x) \) over the interval \([0, 1]\) from a subspace \( V \subset L^p([0, 1]) \): Find \( u^* \in V \) such that

\[
||u - u^*||_{L^p} = \inf_{v \in V} ||u - v||_{L^p},
\]

where

\[
||u||_{L^p} = \left( \int_0^1 |u|^p \, dx \right)^{1/p}, \quad 1 \leq p < \infty, \quad ||u||_{L^\infty} = \sup_{x \in [0, 1]} |u(x)|.
\]

We wish to find the best \( L^p \)-approximation of the specific function \( u(x) = x^4 \).

(a) Determine the best \( L^2 \)-approximation in the subspace of quadratic functions; i.e., \( V = \text{span}\{1, x, x^2\} \), and justify the technique you use.

(b) Why (specifically) does this problem become tremendously more difficult if we consider the case \( p \neq 2 \)?

(c) Prove that the decomposition of an element of a Hilbert space using the Projection Theorem is unique.