Let $p$ be a prime, and $G$ a group of order $p^3$. Prove that $G$ has a normal subgroup of order $p^2$.

"[n]" means the problem is worth $n$ points.

1. Let $p$ be a prime, and $G$ a group of order $p^3$. Prove that $G$ has a normal subgroup of order $p^2$. 

Name: ____________________________
b. Assume that $G$ has a cyclic normal subgroup $N$ of order $p^2$, generated by some element $n$. Let $g$ be an element not in $N$.
i [5]. If the order $|g|$ of $g$ is $p^2$, classify the possible $G$ up to isomorphism.

ii [15]. If the order $|g|$ of $g$ is $p$, classify the possible $G$ up to isomorphism.

(Incidentally, there exist groups of neither type, such as the group of $3 \times 3$ upper triangular matrices over $\mathbb{F}_p$ with 1s on the diagonal.)
2. Let $I, J$ be two ideals in a commutative ring $R$ (with unit). Define $K = \{ r : rJ \leq I \}$. Show that $K$ is an ideal.
b [10]. If $R$ is a principal ideal domain, so $I = \langle i \rangle, J = \langle j \rangle$, give a formula for a generator $k$ of $K$. 
Describe, up to isomorphism, all the $\mathbb{R}[t]$-module structures one might put on a 3-dimensional real vector space (extending the fixed $\mathbb{R}$-action).
4. Let $\mathbb{C}[x]/\langle x^n \rangle$ denote the evident $\mathbb{C}[x]$ (bi)module, and let $m, n \in \mathbb{N}$. Show that there exist $d_1, \ldots, d_k$ such that

$$\mathbb{C}[x]/\langle x^n \rangle \otimes_{\mathbb{C}[x]} \mathbb{C}[x]/\langle x^m \rangle \cong \bigoplus_{i=1}^{k} \mathbb{C}[x]/\langle x^{d_i} \rangle.$$
b [20]. Determine the \( \{d_i\} \) in terms of \( m, n \).

*Hint:* figure out the action of \( x \) on the obvious \( C \)-basis.
Recall that a "perfect" field of characteristic $p$ is one for which the Frobenius map $F: x \to x^p$ is onto.

Let $K$ be a perfect field, and $F$ an algebraic extension. Show that $F$ is perfect.