Three hour exam. Each question is worth five marks. You can use any standard facts or theorems in your work provided you state clearly that you are doing so. Please try to write good clear mathematics!

1. Consider a solid tetrahedron $ABCD$. The face $ABC$ is glued to $ABD$ by an affine map preserving the order of vertices (i.e. $A$ goes to $A$, $B$ goes to $B$, $C$ goes to $D$.) Similarly, $BCD$ is glued to $ACD$. Compute the fundamental group of the resulting quotient space.

2. Let $X_n$ be the bouquet of $n$ circles, whose fundamental group (based at the vertex of the bouquet) is the free group $F_n$ on $n$ generators.

   (i). Draw a covering of $X_3$ by $X_5$. Find the subgroup of $F_3 = \langle a, b, c \rangle$ to which your cover corresponds under the correspondence between subgroups of $F_3$ and based connected covers of $X_3$.

   (ii). Show that $X_3$ cannot cover $X_5$.

3. Compute the integral homology $H_*(\mathbb{R}P^2 \times \mathbb{R}P^3; \mathbb{Z})$.

4. Let $T \subseteq S^4$ be a (perhaps knotted) subspace homeomorphic to the 2-torus. Let $W$ be a closed regular neighbourhood of $T$, so that $W$ is homotopy equivalent to $T$. Let $X$ be $S^4$ minus the interior of $W$, so that $X$ is a compact 4-manifold with boundary. By considering the relative cohomology $H^*(S^4, W)$ and applying excision and Lefschetz duality, calculate the homology of $X$.

5. On any closed surface $\Sigma_g$ of genus $g \geq 1$, it is possible to find a pair of simple closed curves (submanifolds homeomorphic to $S^1$) meeting transversely once. Use this fact together with intersection theory to show that any map $S^2 \to \Sigma_g$ has degree zero.

6. Use the Hurewicz theorem to calculate $\pi_3(\mathbb{R}P^3 \vee S^3)$.

7. Show that any closed (i.e. compact, without boundary) 6-manifold which is 2-connected (i.e. path-connected, simply-connected and has $\pi_2 = 0$) must have even Euler characteristic.

8. Let $M^3$ be a homology sphere – a closed 3-manifold having the same homology groups as $S^3$ – and let $X = \Sigma M$ be its suspension. What are the fundamental group and homology groups of $X$? Show that $X$ is homotopy equivalent to $S^4$. 