(25) 1. State and prove the SVD Existence Theorem (for real $m \times n$ matrices).

(20) 2. Let $A$ be the $m \times n$, rank($A$) = $r$. Use the SVD of $A$, $U \Sigma V^T$, to show:
   (a) Nullspace ($A$) = span{$v_{r+1}, \ldots, v_n$}
   (b) Range ($A$) = span{$u_1, \ldots, u_r$}

(30) 3. (a) Let $D$ be an $m \times n$ diagonal matrix. Prove $\|D\|_p = \max |d_{ii}|$ for $1 \leq p \leq \infty$.
   (b) Prove that if $A$ is $m \times n$, rank($A$) = $n$ and $\|E\|_p \|A^{-1}\|_p < 1$ for some $p$, $1 \leq p \leq \infty$, then rank($A + E$) = $n$.
   (c) Let $A$ be $n \times n$, nonsingular, and $A = QR$, where $Q$ is orthogonal and $R$ is upper triangular with positive diagonal. Prove that $Q$ and $R$ are unique.

(30) 4. Suppose the computed $a_{ik}^{(k)} = 0$ for $1 \leq k \leq n - 1$, where $A$ is $n \times n$, then the computed $L$ and $U$ satisfy $A + E = LU$, where $L$ is unit lower triangular and $U$ is upper triangular. Derive the bound on $E$:

$$|E_{ij}| \leq \left\{ \begin{array}{ll}
(3 + u)(i-1)^r |u| & \text{for } i \leq j \\
(3 + u)(j-1)^r |u| & \text{for } i > j
\end{array} \right.$$ 

and $u$ = unit roundoff.

(55) 5. (20) (a) Show that if the single shift $QR$ method converges, then the convergence is: (a) quadratic for general matrices. (b) cubic for symmetric matrices.

(25) (b) Let $A_0 = A$, where $A$ is symmetric positive definite.

   for $k = 1, 2, \ldots$

   $A_{k-1} = G_k G_k^T$ (Cholesky)

   $A_k \equiv G_k^T G_k$

   Prove that if $A = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$ with $a \geq c$ then $A_k = \text{diag}(\lambda_1, \lambda_2)$, where $\lambda_1 \geq \lambda_2 > 0$.

(10) (c) Let $S = \begin{bmatrix} 0 & -B^T \\ B & 0 \end{bmatrix}$, where $B$ is $n \times n$. Relate the eigenvalues and eigenvectors of $S$ to the SVD of $B$, $B = U \Sigma V^T$. 

1
NA Qual. Part C: Approximation, Interpolation, and Numerical Quadrature.

Question 3.1. [20 points]

(1) Let \( f \in C[-1,1] \) be an even function. Let \( p_n \in \mathcal{P}_n \) be the best uniform approximation of \( f \) in \( \mathcal{P}_n \). Prove that \( p_n \) is also an even function.

(2) Let \( n \geq 1 \) be an integer. Let \( l_0(x), \ldots, l_n(x) \) be the Lagrange basic interpolation polynomials associated with \( n + 1 \) distinct points \( x_0, \ldots, x_n \), i.e.,

\[
l_k(x) = \prod_{\substack{j=1, j\neq k \atop j=1, j\neq n}}^{n} \frac{x-x_j}{x_k-x_j}, \quad k = 0, \ldots, n.
\]

Prove that

\[
x^n = \sum_{j=0}^{n} x_j^m l_j(x), \quad m = 1, \ldots, n.
\]

Question 3.2. [20 points]

Let \( n \geq 1 \) be an integer and \( -\infty < a < b < \infty \). Consider the numerical quadrature

\[
\int_a^b f(x) \, dx \approx \sum_{k=1}^{n} A_k f(x_k),
\]

where \( x_1, \ldots, x_n \in [a,b] \) are distinct points and \( A_1, \ldots, A_n \in \mathbb{R} \). Let \( m \) denote the degree of precision of this numerical quadrature. Prove the following:

(i) \( m \leq 2n - 1 \);

(ii) If this is an interpolatory quadrature, then \( m \geq n - 1 \);

(iii) That \( m = 2n - 1 \) if and only if this is a Gaussian quadrature.