Problem 1

Let $X_1, \ldots, X_n$ be iid $\text{Exp}(\theta)$.

1. Identify the exact distribution of $\bar{X} = n^{-1} \sum_{i=1}^{n} X_i$.

2. Find a $(1 - \alpha)100\%$ UMA upper confidence bound for $\theta$ by inverting the appropriate UMP one-sided test.

3. Identify the large-sample distribution of $\bar{X}$.

4. Find a variance stabilizing transformation for $\bar{X}$, and call this $h(\cdot)$.

5. Find an approximate $(1 - \alpha)100\%$ upper confidence bound for $\theta$ from the large-sample distribution of $h(\bar{X})$.

6. Which of the two confidence bounds would you prefer (the one from part 2 or part 5) and why?

7. Is the transformation $h(\cdot)$ also normalizing? If not, can you find a normalizing transformation? (HINT: try a power-law transformation, and use the fact that

$$E[\sqrt{n}(h(\bar{X}) - h(\theta))]^3 \approx \left(|h'(\theta)|^3 \mu_3 + 3h''(\theta)|h'(\theta)|^2 \mu_2^2\right) / \sqrt{n} + O(1/n)$$

where $\mu_k$ is the $k$th central moment of $X_1$.}
Problem 2

1. Let $X_1, ..., X_n$ be iid from a strictly increasing, continuous cdf $F$, and let $Y_i = F(X_i)$ for $i = 1, ..., n$. Show that the common distribution of the $Y_i$ is Uniform (0,1).

2. Can you relax the strictly increasing assumption to just continuity of $F$ in part (a)?

3. Use the result of part (a) to show that the $P$-value of a general test of a point null hypothesis is uniformly distributed under the null. You may assume that the test is based on a test statistic $T$ that has a strictly increasing, continuous cdf $F$. Suppose also (for simplicity) that the rejection region is of the type $T >$ some threshold $t$. 
Problem 3

Let $X_1, \ldots, X_n$ be i.i.d $U(\xi - \theta, \xi + \theta)$, where $\xi \in \mathbb{R}$ and $\theta > 0$ are both unknown.

1. Show that $X_{(1)}$ and $X_{(n)}$ are sufficient statistics and find their (marginal) distributions. [Bonus if you find the joint distribution.]

2. Let $Y = (X_{(n)} - X_{(1)})/2$, with density $\psi(y) = 1/\theta \psi(y/\theta)$. where

\[
\psi(y) = n(n-1)(1-y)y^{n-2} \quad \text{on } y \in (0,1)
\]

[Bonus if you prove it.] Show that $\psi, \theta > 0$ has the monotone likelihood ratio property.

3. For fixed $\delta > 0$, consider testing $H: \theta \leq \delta$ versus $K: \theta > \delta$. Show the problem is invariant under the group of transformations

\[(x_1, \ldots, x_n) \rightarrow (x_1 + c, \ldots, x_n + c), c \in \mathbb{R}\]

and that $X_{(1)}$ and $X_{(n)}$ are equivariant.

4. Find the UMPI level $\alpha$ test (be as explicit as possible).
Problem 4

Let $X_1, \ldots, X_m, Y_1, \ldots, Y_n$ be independent with

$$X_i \sim \mathcal{N}(\xi, \sigma_i^2), \quad Y_j \sim \mathcal{N}(\eta, \tau_j^2)$$

The parameters $\sigma_1, \ldots, \sigma_m, \tau_1, \ldots, \tau_n$ are known positive constants satisfying

$$\sum_{i=1}^m \frac{1}{\sigma_i^2} = \sum_{j=1}^n \frac{1}{\tau_j^2}$$

Consider testing $H : \eta \leq \xi$ versus $K : \xi > \eta$.

1. Show that the following are sufficient statistics and find their joint distribution:

$$U = \sum_{i=1}^m \frac{X_i}{\sigma_i^2}, \quad V = \sum_{j=1}^n \frac{Y_j}{\tau_j^2}$$

2. Find the UMP level $\alpha$ test (be as explicit as possible).

(Hint: for a particular alternative $\xi_1 > \eta_1$, the distribution assigning probability one to $\xi = \eta = (\xi_1 + \eta_1)/2$ is least favorable.)