1. (20pts.) Calculate the fundamental group of $\mathbb{RP}^2 \# \mathbb{RP}^2$.

2. (20pts.) Given a topological space $X$, let $I$ denote the directed set of compact subsets of $X$ under inclusion. Show that the following canonical map is an isomorphism:

$$\text{colim}_{K \in I} H_i(K; R) \longrightarrow H_i(X; R),$$

for any ring $R$.

3. (20pts.) Show that there is no continuous map $f$ with the following properties:

$$f : S^n \longrightarrow S^m, \quad f(-x) = -f(x), \quad n > m \geq 1,$$

where $-x$ denotes the antipode of $x$.

4. (20pts.) Let $M^n$ and $N^n$ be compact, oriented, boundaryless manifolds of dimension $n$ with fundamental classes $\mu_M$ and $\mu_N$ respectively. Assume that $f : M^n \rightarrow N^n$ is a continuous map, and $U \subseteq N^n$ is an open set with the property that $f^{-1}(U) = \bigcup_{i=1}^k U_i$ is a finite disjoint union, such that $f$ restricts to an orientation preserving homeomorphism $f : U_i \rightarrow U$. Show that $f_*(\mu_M) = k \mu_N$.

5. (10pts.) Given a compact, oriented, boundaryless $n$-manifold $M^n$, show that there always exists a continuous map $f : M^n \rightarrow S^n$, such that $f_*(\mu_M) = \mu_{S^n}$.

6. (10pts.) Calculate the integral homology of any compact, oriented, boundaryless manifold $N^n$, which admits a continuous map $f : S^n \rightarrow N^n$ with $f_*(\mu_{S^n}) = \mu_N$.

7. (20pts.) Let $T^2 = S^1 \times S^1$ be the 2-torus. Let $f : T^2 \rightarrow T^2$ be a self-map of degree 1. Show that the map $f^* : H^1(T^2, \mathbb{Z}) \rightarrow H^1(T^2, \mathbb{Z})$ defines an element of $\text{SL}_2(\mathbb{Z})$. Here, we identify $H^1(T^2, \mathbb{Z})$ with $\mathbb{Z} \oplus \mathbb{Z}$ using the identification $T^2 = S^1 \times S^1$. 