Ph.D./Masters Qualifying Examination in Numerical Analysis

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1-4pm
Friday May 22, 2009
6402 AP&M

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- Add your name in the box provided and staple this page to your solutions.
- Write your name clearly on every sheet submitted.
1. Norms, Condition Numbers and Linear Equations

Question 1.1.

(a) Let $u$ denote the unit roundoff, and assume that $nu < 1$ for the positive integer $n$. If $\{\delta_i\}$ are $n$ scalars such that $|\delta_i| \leq u$, prove that

$$\prod_{i=1}^{n} (1 + \delta_i) = 1 + \theta_n,$$

where $|\theta_n| \leq \gamma_n$,

with $\gamma_n = nu/(1 - nu)$.

(b) State the standard rounding-error model for floating-point arithmetic. Given representable numbers $a$ and $b$, compute the backward and forward relative error for the floating-point value $s$ of the expression $s = \sqrt{a^2 + b^2}$. (You may assume that the square root function conforms to the standard rounding-error model for floating-point arithmetic.)

Question 1.2.

(a) Define the spectral condition number $\text{cond}_2(A)$ for any $A \in \mathbb{R}^{m \times n}$. State (but do not prove) an expression for $\text{cond}_2(A)$ in terms of the singular values of $A$.

(b) Let $A \in \mathbb{R}^{n \times n}$ be nonsingular. Find a solution of the problem

$$\min_{E \in \mathbb{R}^{n \times n}} \{\|E\|_2 : A + E \text{ singular}\}.$$

(c) Prove that $1/\text{cond}_2(A)$ is the relative two-norm distance of $A$ to the nearest singular matrix.

Question 1.3. Assume that $A$ is an $m \times n$ matrix with rank $k$ ($k < \min(m, n)$).

(a) Define what is meant by a full-rank factorization $A = BC$.

(b) State the full-rank factorization of $A$ in terms of the QR decomposition with column interchanges. (You may assume that the decomposition is computed in exact arithmetic.)

(c) Using the QR decomposition of part (b), define bases for the subspaces $\text{range}(A)$ and $\text{null}(A)$. Prove that the proposed bases satisfy the properties of a subspace basis.

(d) Using the $QR$ decomposition of part (b), define orthogonal projections onto $\text{range}(A)$ and $\text{null}(A)$. Prove that the proposed projections satisfy the properties of an orthogonal projection.

(e) Derive the pseudoinverse of $A$ in terms of the full-rank factorization of part (b).
2. Nonlinear Equations and Optimization

Question 2.1. Let $A$ denote an $n \times n$ matrix, and let $s$ and $y$ be arbitrary $n$-vectors.

(a) Find all the eigenvalues of the matrix $I + \gamma uv^T$, where $\gamma$ is a scalar and $u$ and $v$ are $n$ vectors.

(b) Consider the Broyden update formula

$$A_+ = A + \frac{1}{s^Ts}(y - As)s^T.$$ 

If $\| \cdot \|_F$ denotes the Frobenius norm, show that $A_+$ minimizes $\min \|B - A\|_F$ over all $B$ such that $Bs = y$.

(c) If $A$ is nonsingular, find a condition on $A$, $s$ and $y$ that will ensure that $A_+$ is nonsingular.

Question 2.2. Let $f : D \subseteq \mathbb{R}^n \to \mathbb{R}$ be twice differentiable on an open convex set $D_0 \subseteq D$.

(a) State the first- and second-order necessary conditions for $x^* \in \mathbb{R}^n$ to be an unconstrained minimizer of $f$.

(b) State first and second-order sufficient conditions for $x^* \in \mathbb{R}^n$ to be an unconstrained minimizer of $f$.

(c) Consider the function $f : \mathbb{R}^3 \to \mathbb{R}$ such that

$$f(x) = (x_3 - 1)^2 \sin x_1 + x_1^2 + x_2^2 - \pi x_1.$$ 

(i) Write down the quadratic model $q(x)$ that interpolates $f$, $\nabla f$ and $\nabla^2 f$ at the point $x_0 = (-\pi/2, 0, \pi + 1)^T$.

(ii) Find the step $p_N$ from $x_0$ to a stationary point of the quadratic model.

(iii) Determine if $p_N$ is a descent direction for $f$ at $x_0$.

(iv) Find a descent direction of negative curvature at $x_0$ (if one exists).

Question 2.3. Given an $n \times n$ symmetric matrix $B$ and vectors $y$ and $s$, consider the symmetric rank-one formula

$$B_+ = B + \frac{1}{(y - Bs)^T(y - Bs)}(y - Bs)(y - Bs)^T.$$ 

(a) Let $f(x)$ be a quadratic function with positive-definite Hessian $H$. Let $s = x_+ - x$ and $y = \nabla f(x_+) - \nabla f(x)$. If vectors $\tilde{s} = \tilde{x}_+ - \tilde{x}$ and $\tilde{y} = \nabla f(\tilde{x}_+) - \nabla f(\tilde{x})$ satisfy $B\tilde{s} = \tilde{y}$, show that $B_+ \tilde{s} = \tilde{y}$.

(b) Show that if $B$ is symmetric and positive definite, then $B_+$ will be positive definite if and only if

$$\frac{y^TB^{-1}y - y^Ts}{y^Ts - s^TBs} > 0.$$
3. Approximation and Numerical ODEs

In this part, we assume that $a, b \in \mathbb{R}$ with $a < b$. We also denote by $\mathcal{P}_n$ the set of all polynomials of degree $\leq n$ for any integer $n \geq 0$.

Question 3.1.

(a) Let $n \geq 0$ be an integer and $T_n$ the $n$th Chebyshev polynomial of first kind. Let $P \in \mathcal{P}_n$ satisfy that $|P(x)| \leq 1$ for all $x \in [-1, 1]$. Show that

$$|P(y)| \leq |T_n(y)| \quad \forall y \in [-1, 1].$$

(b) Let $\mathcal{F}$ denote the class of functions $a_0 + a_1 \cos x + a_2 \cos 2x$ with $a_0, a_1, a_2 \in \mathbb{R}$. Find $T \in \mathcal{F}$ such that

$$\int_{-1}^{1} |T(x) - x|^2 dx \leq \int_{-1}^{1} |S(x) - x|^2 dx \quad \forall S \in \mathcal{F}.$$

Question 3.2.

(a) Use the error formula for the Lagrange interpolation of $f \in C^2[a, b]$ at the two points $a$ and $b$ to derive the error for the trapezoidal numerical integration rule

$$\int_a^b f(x) \, dx \approx \frac{1}{2} (b - a) [f(a) + f(b)].$$

(b) Assume that $f \in C^2[a, b]$. Derive an error formula for the composite trapezoidal numerical integration rule

$$\int_a^b f(x) \, dx \approx \frac{h}{2} [f(a) + f(b)] + h \sum_{j=1}^{N-1} f(x_j).$$

Here $N \geq 1$ is an integer, $h = (b - a)/N$, and $x_j = a + jh$ ($j = 0, \ldots, N$).

(c) Apply the composite trapezoidal numerical integration rule to

$$\int_0^{10} \sin x \, dx.$$

How large $N$ is needed so that the error of the numerical integration is less than $10^{-6}$? (Ignore the round-off error.) Justify your answer.
3. Approximation and Numerical ODEs

In this part, we assume that $a, b \in \mathbb{R}$ with $a < b$. We denote by $\mathcal{P}$ the set of all real polynomials. For any integer $n \geq 0$, we denote by $\mathcal{P}_n$ the set of all real polynomials of degree $\leq n$.

Question 3.1.

(a) Let $f \in C[a, b] \cap \mathcal{P}$. For any integer $n \geq 0$, denote

$$E_n(f) = \min_{p_n \in \mathcal{P}_n} \max_{a \leq x \leq b} |f(x) - p_n(x)|.$$ 

Prove that the sequence $\{E_n(f)\}_{n=0}^\infty$ is strictly decreasing i.e., $E_n(f) > E_{n+1}(f)$ for all $n \geq 0$ and converges to 0.

(b) Find the real numbers $A$, $B$, and $C$ so that the numerical quadrature

$$\int_{-2}^{2} f(x) \, dx \approx Af(-1) + Bf(0) + Cf(1)$$

has the highest possible degree of precision. What is this highest possible degree of precision?

Question 3.2. Let $x_1^{(a)}, \ldots, x_n^{(a)}$ be the $n$ distinct roots of orthogonal polynomials $Q_n$ in $L^2(a, b)$ ($n = 1, 2, \ldots$).

(a) For each $n \geq 2$, let $l_1^{(n)}, \ldots, l_n^{(n)}$ be the Lagrange basis polynomials associated with $x_1^{(a)}, \ldots, x_n^{(a)}$. Prove the following:

$$\int_a^b l_j^{(n)}(x) l_k^{(n)}(x) \, dx = 0 \quad \text{if } 1 \leq j, k \leq n, \text{ and } j \neq k;$$

$$\sum_{k=1}^n \int_a^b [l_k^{(n)}(x)]^2 \, dx = b - a.$$ 

(b) Let $L_{n-1} : C[a, b] \to \mathcal{P}_{n-1}$ be the Lagrange interpolation operator associated with $x_1^{(n)}, \ldots, x_n^{(n)}$. Prove that

$$\lim_{n \to \infty} \int_a^b [f(x) - (L_{n-1}f)(x)]^2 \, dx = 0 \quad \forall f \in C[a, b].$$