1. (100 pts.) For each of the following, determine if the statement is always true or if it is false. If true, give a proof. If false, give a counterexample or disprove it. Be brief!
(a) Let $G = \{ z \in \mathbb{C} : \text{Re } z \notin \mathbb{Z} \}$. Suppose $f \in H(G)$ such that $|f(z)| \leq 1$ for all $z \in G$. Then $f$ is constant.
(b) There exists $g \in H(B(0; 2))$ with $|g(z)| < 1$ for all $|z| < 2$, with $z + g(z) \neq 0$ for all $z \in D$.
(c) There is an analytic function $f$ on $D \setminus \{ 0 \}$ with an essential singularity at $z = 0$ such that $f$ can be extended as a continuous function from the whole disk $D$ to the extended plane $\mathbb{C}_\infty$.
(d) Suppose that $f$ is a continuous function on $\overline{D} \cap \{ z : \text{Im } z \leq 0 \}$ and analytic in $\overline{D} \cap \{ z : \text{Im } z < 0 \}$. If $\text{Re } f(x) = 0$ for all $x$ with $-1 < x < 1$, then $f$ admits an analytic extension to $D$.
(e) If $G \subset \mathbb{C}$ is a region and $G \cap B(0; r) = \emptyset$ for some $r > 0$, there is a $1 - 1$ analytic function $f$ on $G$ such that $f(G) \subset D$.
(f) If $f : \mathbb{C} \to \mathbb{C}$ is analytic function and $\text{Re } f(z) \geq c$ for some real constant $c$, then $f$ is constant.
(g) There is a polynomial $p(z)$ such that $|p(z) - 1/z| < 1$ for all $z$ in the annulus $1/2 < |z| < 3/2$.

2. (25 pts.) Let $G$ be a bounded region and
\[ \mathcal{F} = \{ f \in C(\overline{G}, \mathbb{C}) : f \text{ analytic in } G \}, \]
and put $\| f \| = \sup_{z \in \partial G} |f(z)|$. Show that $\mathcal{F}$ is a complete metric space with $d(f, g) = \| f - g \|$. (Here $\overline{G}$ is the closure of $G$ in $\mathbb{C}$, $C(\overline{G}, \mathbb{C})$ is the space of all complex-valued, continuous functions in $\overline{G}$, and $\partial G$ is the boundary of $G$.)
3. (25 pts.) Let $a = 1$, $b = 0$, regarded as points in $\mathbb{C}$.
(a) Give an example of an analytic function element $(h, D)$, with $a \in D$, and analytic continuations $(f_t, D_t)$, $(g_t, B_t)$ of $(h, D)$ along two different paths $\gamma$ and $\sigma$, such that $\gamma(0) = \sigma(0) = a$, $\gamma(1) = \sigma(1) = b$, but $[g_1]_b \neq [f_1]_b$. (Note that necessarily you have $[g_0]_a = [f_0]_a = [h]_a$.) Define the analytic continuations explicitly (i.e. be sure to define $D_t$ and $B_t$ as well as the germs $[f_t]_{\gamma(t)}$, $[g_t]_{\sigma(t)}$) and explain briefly why these satisfy the definition of an analytic continuation.
(b) Explain why the Monodromy Theorem does not apply for your example in (a) to conclude that $[g_1]_b = [f_1]_b$.

4. (25 pts.) (a) Let $(X, d)$ be a metric space, $\{x_n\}$ a sequence in $X$, and $x \in X$. Suppose that every subsequence of $\{x_n\}$ has a subsequence which converges to $x$. Show that $\{x_n\}$ converges to $x$.
(b) Let $\{f_n\}$ be a sequence of locally bounded analytic functions in an open region $G \subset \mathbb{C}$. Let $A := \{z \in G \colon \lim_{n \to \infty} f_n(z) = 0\}$ and assume that $A$ has a limit point in $G$. Show that $\{f_n\}$ converges uniformly on compact subsets of $G$ to $f \equiv 0$.

5. (25 pts.) For $R > 1$ let $C_R$ be the quarter circle parametrized by $z = Re^{i\theta}$, $0 \leq \theta \leq \pi/2$. Prove that
$$\lim_{R \to \infty} \int_{C_R} \frac{e^{iz}}{\log z} \, dz = 0,$$
where $\log z$ is the principal branch of the logarithm. [Hint: You may use the inequality $\sin \theta \geq c\theta$ for $0 \leq \theta \leq \pi/2$, with some constant $c > 0$. Then evaluate an appropriate integral.]