Applied Algebra Qualifying Exam: Part I

9:00am–Noon, AP&M 6402
Tuesday May 28th, 2013

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- Do all four problems.
- This part of the exam will represent 40% of your total score.
- Add your name in the box provided and staple this page to your solutions.

- Notation:
  - $\mathcal{M}_{m,n}$ denotes the set of $m \times n$ matrices with complex entries.
  - If $m = n$, $\mathcal{M}_{m,n}$ is denoted by $\mathcal{M}_n$.
  - $\mathbb{C}^n$ is the set of column vectors with $n$ complex entries.
  - $x^H$ is the Hermitian transpose of a vector or matrix $x$.
  - $\text{eig}(A)$ is the set of eigenvalues of the matrix $A$ (counting multiplicities).
  - $\text{Re}(\lambda)$ and $\text{Im}(\lambda)$ denote the real and imaginary parts of the scalar $\lambda$. 
Question 1.

(a) (8 points) Prove the Schur decomposition theorem for a matrix $A \in M_n$.

(b) (12 points) Prove that for $A, B \in M_n$, if $x^H A x = x^H B x$ for all $x \in \mathbb{C}^n$, then $A = B$.

Question 2.

(a) (10 points) Prove that every $A \in M_n$ may be written uniquely as $A = S + iT$, where $S$ and $T$ are Hermitian.

(b) (10 points) For any $A \in M_n$, consider the unique expansion $A = S + iT$, where $S$ and $T$ are Hermitian. Prove that for any $\lambda \in \text{eig}(A)$, it holds that

$$\lambda_n(S) \leq \text{Re}(\lambda) \leq \lambda_1(S) \quad \text{and} \quad \lambda_n(T) \leq \text{Im}(\lambda) \leq \lambda_1(T),$$

where, by convention, the eigenvalues of a Hermitian matrix $C \in M_n$ are arranged in nonincreasing order, i.e.,

$$\lambda_1(C) \geq \lambda_2(C) \geq \cdots \geq \lambda_n(C).$$

Question 3.

(a) (4 points) Define the $p$-norm $\|A\|_p$ and Frobenius norm $\|A\|_F$ of a matrix $A \in M_{m,n}$.

(b) (10 points) Suppose that $D \in M_n$ with $D = \text{diag}(d_1, d_2, \ldots, d_n)$. Prove that for all $1 \leq p \leq \infty$ the $p$-norm of $D$ is given by $\|D\|_p = \max_{1 \leq i \leq n} |d_i|$.

(c) (6 points) Given $b \in \mathbb{C}^{n-1}$, find $\|B\|_2$ for the matrices

$$B = \begin{pmatrix} 0 & b^H \\ b & 0 \end{pmatrix} \quad \text{and} \quad B = bb^H.$$

(Show your work. Simply writing down the answer will not be sufficient.)

Question 4.

(a) (15 points) Prove that if $A \in M_n$ is positive semidefinite, then there exists a unique positive semidefinite $X$ such that $A = X^2$.

(b) (5 points) Let $X$ be a matrix whose columns define a basis for a subspace $X \subset \mathbb{C}^n$. Consider the matrix $\hat{X} = X|X|^{-1}$, where $|X|$ denotes the modulus of $X$, i.e., $|X| = (X^HX)^{\frac{1}{2}}$. Prove that $\hat{X}$ exists and that $\hat{X}\hat{X}^H$ is an orthogonal projection onto $X$. 
Do as many problems as you can, but you must attempt at least 5 problems where two of the problems are from problems 1-5, one problem for 6-7, and one problem are from problems 8-9. The point values are relative values for this part of the exam. Your final score will be scaled so that this part of the exam will represent 60% of your point total.

Let $\mathbb{N} = \{0, 1, 2, \ldots\}$, $\mathbb{Z} = \{0, \pm 1, \pm 2, \ldots\}$, $\mathbb{Q}$ equal the rationals and $\mathbb{C}$ denote the complex numbers.

Suppose that $\lambda = (\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_k)$ is a partition of $n$. Then $A^\lambda$ denotes the irreducible representation of the symmetric group $S_n$ such that the Frobenius image of $\chi^A = \chi^\lambda$ is the Schur function $S_\lambda(x_1, \ldots, x_N)$ where $N > n$ and $S_{\lambda_1} \times \cdots \times S_{\lambda_k}$ denotes the Young subgroup of $S_n$ corresponding to $\lambda$.

(1) (30 pts.) Let $H$ be a subgroup of $G$ and $A : H \rightarrow \text{GL}_n(\mathbb{C})$ be a representation of $H$. Let $\chi^A : H \rightarrow \mathbb{C}$ be the character of $A$. Define $\chi^A : G \rightarrow \mathbb{C}$ by

$$\chi^A(\sigma) = \begin{cases} \chi^A(\sigma) & \text{if } \sigma \in H \text{ and } \\ 0 & \sigma \in G - H. \end{cases}$$

(a) Define the representation $A \uparrow^G_H$.

(b) Prove that $\chi^A \uparrow^G_H = \frac{1}{|H|} \sum_{\sigma \in G} \sigma \cdot \chi^A \cdot \sigma^{-1}$.

(c) State and prove the Frobenius Reciprocity Theorem.

(2) (40 pts)

(a) Compute the values of the character $\chi^{(1,2^2)}$ on the conjugacy classes of $S_5$.

(b) Find the character table of $S_3 \times S_2$.

(c) Decompose the $A^{(1,2^2)} \downarrow^{S_5}_{S_3 \times S_2}$ as a sum of irreducible characters of $S_3 \times S_2$.

(3) (40 pts) Let $Q$ be the quaternion group of order 8 defined by the relations

$$a^4 = 1, \quad a^2 = b^2, \quad \text{and} \quad b^{-1}ab = a^3.$$ 

(a) Show that $ba = ab^3 = a^2b$ and, hence, that every element of $Q$ is of the form $a^i$ or $a^i b$ for some $i \in \{0, 1, 2, 3\}$.

(b) Verify that the conjugacy classes of $G$ are $C_1 = \{1\}$, $C_2 = \{a^2\}$, $C_3 = \{a, a^3\}$, $C_4 = \{b, a^2b\}$, and $C_5 = \{ab, a^3b\}$.

(c) Show that $H = \{1, a^2\}$ is a normal subgroup of $G$ for which $G/H$ is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$.

(d) Give the character character table for the lifting of the four linear characters of $Q/H$ to $Q$.

(e) Use parts (c) and (d) to give the complete character table for $Q$. 

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(4) (30 pts)

(a) Let \( T \) denote the trivial representation on the Young subgroup \( S_2 \times S_3 \times S_1 \) of \( S_6 \) and \( Alt \) denote the alternating representation on the Young subgroup \( S_2 \times S_3 \times S_1 \) of \( S_6 \). Express the characters of \( T \uparrow_{S_2 \times S_3 \times S_1}^{S_6} \) and \( Alt \uparrow_{S_2 \times S_3 \times S_1}^{S_6} \) as a sum of irreducible characters of \( S_6 \).

(b) Find the decomposition of the Kronecker product \( A^{(1,4)} \otimes A^{(1,2^2)} \) as a sum of irreducible representations of \( S_5 \).

(c) Find the decomposition of \( A^{(1,2)} \times A^{(1,3)} \uparrow_{S_3 \times S_4}^{S_7} \) as a sum of irreducible representations of \( S_7 \).

(5) (40 pts.) Let \( G \) and \( H \) be finite groups and let \( A : G \rightarrow GL_n(C) \) and \( B : H \rightarrow GL_m(C) \) be representations of \( G \) and \( H \) respectively.

a) Show that \( A \times B : G \times H \rightarrow GL_{nm}(C) \) is representation where for \((\sigma, \tau) \in G \times H\),

\[
A \times B((\sigma, \tau)) = A(\sigma) \otimes B(\tau)
\]

and for matrices \( M \) and \( N \), \( M \otimes N \) is the Kronecker product of \( M \) and \( N \).

b) Show that \( A \times B \) is an irreducible representation of \( G \times H \) if and only if \( A \) is an irreducible representation of \( G \) and \( B \) is an irreducible representation of \( H \).

c) Show that every irreducible representation of \( G \times H \) is of the form \( A \times B \) where \( A \) is an irreducible representation of \( G \) and \( B \) is an irreducible representation of \( H \).

(d) Show that it is not always the case that if \( C \) is a representation of \( G \times H \), then \( C \) is similar to a representation of the form \( A \times B : G \times H \rightarrow GL_n(C) \) where \( A \) is representation of \( G \) and \( B \) is representation of \( H \). (Hint: Consider the two dimensional representations of \( S_2 \times S_2 \).)

(6) (40 pts.) Consider the equations

\[
\begin{align*}
x^2 - xy - 2x &= 0 \\
y^2 - 2xy - y &= 0
\end{align*}
\]

(a) Let \( I \) be the ideal of \( \mathbb{C}[x, y] \) generated by these equations. Find the reduced Groebner basis for \( I \) relative to lexicographic order where \( y > x \).

(b) Find a reduced Groebner basis for \( \mathbb{C}[x] \cap I \).

(c) Find all solutions to these equations that lie \( \mathbb{C}^2 \).

(d) Find a vector space basis for \( \mathbb{C}[x, y]/I \).
(7) (30 pts.) Let $S$ be the parametric surface defined by
\[
    \begin{align*}
        x &= u - 2v \\
        y &= uv \\
        z &= v 
    \end{align*}
\]
(a) Compute a reduced Groebner basis for the ideal generated by this set of equations relative to the lexicographic order where $u > v > x > y > z$.

(b) Find the equation of the smallest variety $V$ that contains $S$.

(c) Show that $S = V$.

(8) (40 pts.) Let $k$ be an algebraically closed field.
Two ideals $I$ and $J$ of $k[x_1, \ldots, x_n]$ are said to be *comaximal* if and only if $I + J = k[x_1, \ldots, x_n]$.

(a) State the Weak Nullstellensatz and Hilbert’s Nullstellensatz Theorem.

(b) Show that two ideals $I$ and $J$ are comaximal if and only if $V(I) \cap V(J) = \emptyset$.

(c) Show that if $I$ and $J$ are ideals in $k[x_1, \ldots, x_n]$, then $I \cap J = (tI + (1-t)J) \cap k[x_1, \ldots, x_n]$.

(d) Show that if $I = \langle f \rangle$ and $J = \langle f \rangle$, then $I \cap J = \langle h \rangle$ where $h$ is a least common multiple of $f$ and $g$.

(9) (30 pts.) Let $A = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}$.

(a) Show that $A$ generates a matrix group $G$ of order three.

(b) Find a set of homogeneous $G$-invariant polynomials which generate $\mathbb{C}[x, y]^G$.

(c) Compute the Hilbert Series of $\mathbb{C}[x, y]^G$. 

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