# Ph.D./Masters Qualifying Examination
in Numerical Analysis

Examiners: Randolph Bank and Philip E. Gill

9:00am–12:00pm
Friday May 21, 1999

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- Add your name in the box provided and staple this page to your solutions.
- Write your name clearly on every sheet submitted.
1. Linear Equations and Linear Least Squares

Question 1.1. Let $\hat{x}$ be an approximate solution of $Ax = b$, where $A$ is a nonsingular $n \times n$ matrix, $n > 1$.

(a) Prove that an infinite number of matrices $F$ satisfy the relation $(A + F)\hat{x} = b$.

(b) Find a matrix $E$ that has the smallest two-norm among all matrices $F$ satisfying $(A + F)\hat{x} = b$.

(c) Show that for the matrix $E$ of part (b),
\[
\frac{\|r\|}{\|b\|} \geq \frac{\|E\|/\|A\|}{1 + \|E\|/\|A\|},
\]
where all norms are two-norms. Hence show that if
\[
\frac{\|r\|}{\|b\|} \leq \epsilon < 1, \quad \text{then} \quad \frac{\|E\|}{\|A\|} \leq \frac{\epsilon}{1 - \epsilon}.
\]
Briefly state the significance of this result when solving linear systems using Gaussian elimination.

Question 1.2.

(a) Let $u$ denote the unit roundoff and assume that $nu < 1$ for some positive integer $n$. If $|\delta_i| \leq u$, show that
\[
\prod_{i=1}^{n} (1 + \delta_i) = 1 + \theta_n,
\]
where $|\theta_n| \leq \gamma_n = nu/(1 - nu)$.

(b) Consider the matrix-vector product $y = Ax$ for $A \in \mathbb{R}^{m \times n}$. Assuming the standard model for floating-point computation, let $\hat{y}$ denote the computed value of $y$ when $A$ and $x$ are representable. Derive the following:

(i) A bound on the Frobenius norm of the absolute backward error in $\hat{y}$, assuming that $A$ is data and $x$ is exact.

(ii) A bound on the two-norm of the absolute forward error in $\hat{y}$.

(iii) A bound on the two-norm of the relative forward error in $\hat{y}$, assuming that $A$ is nonsingular.

Question 1.3. Assume that $A \in \mathbb{R}^{m \times n}$.

(a) Derive the necessary and sufficient condition for a vector to be a solution of minimum two-norm for the compatible system $Ax = b$.

(b) Show that the least-length solution of the compatible system $Ax = b$ is unique.

(c) For any $x$, let $r$ denote the residual vector $b - Ax$. Derive the necessary and sufficient conditions for a vector to solve the problem $\min \|b - Ax\|_2$. Discuss the circumstances under which the least-squares solution is unique.

(d) If $A \in \mathbb{R}^{n \times n}$ and $x \in \mathbb{R}^n$ are given, find the value $\lambda^*$ that minimizes the function $\|Ax - \lambda x\|_2$. 

2. Eigenvalues and Singular Values

Question 2.1. Prove the existence of the singular-value decomposition for an $m \times n$ complex-valued matrix with $m \geq n$.

Question 2.2. Assume that $A \in \mathbb{C}^{n \times n}$. Given a positive scalar $\epsilon$ and $E \in \mathbb{C}^{m \times n}$ such that $\|E\|_2 = 1$, let $\lambda$ denote an eigenvalue of $A + \epsilon E$.

(a) Show that if $\lambda$ is not an eigenvalue of $A$, then $\lambda$ lies in the domain

\[
\frac{1}{\| (A - \lambda I)^{-1} \|_2} \leq \epsilon.
\]

(b) Show that if $E$ is of the form $-pq^H$, then the value of $\epsilon$ that produces the eigenvalue $\lambda$ is

\[
\epsilon = \frac{1}{q^H (A - \lambda I)^{-1} p}.
\]

(c) Hence find a perturbation that gives an eigenvalue on the boundary of the domain defined in part (a).

Question 2.3. Consider the unshifted QR method for finding the eigenvalues of a matrix $A \in \mathbb{C}^{n \times n}$. If $R_k$ and $Q_k$ are the matrices generated at iteration $k$, show that

(a) $A_{k+1} = Q_k^H A_k Q_k$.

(b) $A_{k+1} = Q_k^H Q_{k-1} \cdots Q_0^H A Q_0 Q_1 \cdots Q_k$.

(c) $A_{k+1} = \tilde{Q}_k \tilde{R}_k$, where $\tilde{R}_k = R_k R_{k-1} \cdots R_1 R_0$ and $\tilde{Q}_k = Q_0 Q_1 \cdots Q_k$.

(d) Define an unshifted QR method that requires $O(n^2)$ floating-point operations each iteration.
3. Interpolation, Approximation and ODEs

Question 3.1. Let $f \in C^2(I)$, $I = [a,b]$, and let $x_i = a + ih$, $0 \leq i \leq n$, $h = (b - a)/n$ be a uniform mesh on $I$. Let $S$ be the space of continuous piecewise linear polynomials with respect to this uniform mesh and let $\tilde{f}$ denote the continuous piecewise linear polynomial interpolant of $f$.

(a) Compute the dimension of $S$ and define the standard nodal basis functions $\{\phi_i\}$ for $S$.

(b) Using the Peano Kernel Theorem, prove:

$$\|f - \tilde{f}\|_{L^2(I)} \leq C h^2 \|f''\|_{L^2(I)}$$

(You do NOT need to explicitly evaluate the constant $C$.)

Question 3.2. Let

$$I(f) = \int_{-1}^{1} f(x)dx$$

Consider the two point Gauss-Legendre quadrature formula of the form

$$Q(f) = w_1 f(x_1) + w_2 f(x_2) \quad (3.1)$$

(a) Find the knots $x_1$ and $x_2$ and the weights $w_1$ and $w_2$ for the Gauss-Legendre formula (3.1).

(b) What is the form of the error $I(f) - Q(f)$? Be sure to explicitly evaluate the constant.

(c) Write down the composite formula for approximating

$$\int_{-1}^{1} f(x)dx$$

on a uniform mesh of size $h$ (note here the reference interval is $[-1, 1]$).

(d) Write down an expression for the error in the composite formula.