(1) Derive the general formula for the expectation of a quadratic form in terms of a random vector where all second moments exist.

Then write down the standard estimator of \( \sigma^2 \) for the (equal variances) linear model and show that this estimator is unbiased.

(2) Given a random sample from the Poisson distribution, denote by \( M \) and \( V \) the sample mean and sample variance, respectively. What is \( E(V|M) \)? (Don't make a major project out of this !)

(3) A discrete random variable \( X \) takes on exactly two values, namely, -1 and +1 where \( P[X=-1] = P[X=+1] = 1/2 \).
   
   a. Show that the mean is 0 and variance is 1.
   b. Show that the characteristic function is given by
      \[ \varphi(t) = \cos(t) \ . \]
   c. Finally, if \( X_1, X_2, \ldots, X_n \) are i.i.d. from this distribution, Use the result of (3) b. to show that
      \[ \sqrt{n} \ \bar{X}_n \xrightarrow{L} Z \sim N(0,1) . \]

(4) Discuss (briefly!) the key results that justify your work in (3) c.

(5) Write down a complete statement of the Central Limit Theorem in the Euclidean Space setting. Then write out a proof (for \( d=1 \)).
(6) Given below is a "Slutsky-like" theorem and its proof. As you read the proof, provide justification for steps (a) through (d).

**Theorem:** Let \( \{X_n, Y_n\} \) be a sequence of pairs of random variables.

\[
P \xrightarrow{L} \implies |X_n - Y_n| \to 0 \quad \text{and} \quad Y_n \xrightarrow{L} Y \implies X_n \xrightarrow{L} Y.
\]

**Proof:** Denote by \( F_{X_n} \) the c.d.f. of \( X_n \) and by \( F_Y \) that of \( Y \). Let \( Y_n - X_n = Z_n \), and let \( x \) be a continuity point of \( F_Y \). Then

\[
F_{X_n}(x) = P(X_n < x) = P(Y_n < x + Z_n)
\]

\[
= P(Y_n < x + Z_n, Z_n < \varepsilon) + P(Y_n < x + Z_n, Z_n \geq \varepsilon)
\]

\[
\leq P(Y_n < x + \varepsilon) + P(Z_n \geq \varepsilon).
\]

Taking limits, we obtain

\[
(b) \quad \limsup_n F_{X_n}(x) \leq F_Y(x + \varepsilon).
\]

Similarly, we have

\[
(c) \quad \liminf_n F_{X_n}(x) \geq F_Y(x - \varepsilon). \quad \text{(Go through the analogue of the development that led to (a) and (b).)}
\]

\[
(d) \quad \text{Finally we obtain } \lim_n F_{X_n}(x) = F_Y(x) \text{ and the proof is complete.}
\]

(7) Use the theorem in Exercise (6) to show that

\[
P \xrightarrow{L} \implies X_n \xrightarrow{L} X_n \xrightarrow{L} X.
\]