Complex Analysis Qualifying Exam – Fall 2016

Name: __________________________________________

Student ID: _________________________________

Instructions:

You have 3 hours. No textbooks and notes are allowed. Make sure to state clearly the hypotheses of any results used.

Solve at least 7 of the following 8 problems. You have 180 minutes to complete the test.

Notation: $\mathbb{D} = \{z \in \mathbb{C} \mid |z| < 1\}$.

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Problem 1. [10 points.]

Prove that if $f$ is an entire function such that $\lim_{z \to \infty} f(z) = \infty$, then $f$ must be a polynomial.
Problem 2. [10 points.]

Assume that $f : \mathbb{D} \to \mathbb{D}$ is an analytic function such that $f(0) = 0$. Show that

$$g(z) = \sum_{n=0}^{\infty} f(z^n)$$

converges to an analytic function on $\mathbb{D}$. 
Problem 3. [10 points.]

Using the calculus of residues, compute

\[ \int_0^\infty \frac{\log x}{x^2 + 1} \, dx. \]
Problem 4. [10 points.]

Let \( f : \mathbb{D} \to \mathbb{C} \) be a continuous function which is analytic on \( \mathbb{D} \). Assume that there exists \( 0 < \alpha \leq 2\pi \) such that \( f(e^{i\theta}) = 0 \), for all \( \theta \in (0, \alpha) \). Prove that \( f(z) = 0 \), for every \( z \in \mathbb{D} \).
Problem 5. [10 points.]

Let $U \subset \mathbb{C}$ be a connected open and let $a \in U$. Let $f_n : U \to \mathbb{D}$ be a sequence of analytic functions such that $f_n(a) = 0$, for all $n \geq 1$. Prove that there exists an analytic function $f : U \to \mathbb{D}$ and a subsequence $\{f_{n_k}\}$ of $\{f_n\}$ which converges uniformly to $f$ on compact subsets of $U$. 
Problem 6. [10 points; 5, 5.]

For $k \geq 1$, let $a_k = 1 - \frac{1}{k^2}$. For $n \geq 1$, define $f_n: \mathbb{D} \to \mathbb{D}$ by letting

$$ f_n(z) = \prod_{k=1}^{n} \frac{a_k - z}{1 - a_k z}. $$

(a) Prove that the sequence $\{f_n\}$ converges to an analytic function $f : \mathbb{D} \to \mathbb{D}$, uniformly on compact subsets of $\mathbb{D}$.

(b) Prove that there do not exist an open set $U \subset \mathbb{C}$ and an analytic function $g : U \to \mathbb{C}$ such that $\overline{\mathbb{D}} \subset U$, and $g(z) = f(z)$, for every $z \in \mathbb{D}$. 
Problem 7. [10 points.]

Let $\gamma : [0,1] \rightarrow \mathbb{C}$ be a path such that $\gamma(0) = 1$ and $\gamma(t) \neq 0$, for every $t \in [0,1]$. Assume that $(f_t, D_t)_{0 \leq t \leq 1}$ is an analytic continuation of $f_0(z) = \log z$ along $\gamma$. Prove that $f_t$ is a branch of the logarithm, for every $t \in [0,1]$. 
Problem 8. [10 points; 5, 5.]

Let $u : \mathbb{C} \to \mathbb{R}$ be a harmonic function such that $\int \int |u(x + iy)|^2 \, dx \, dy < \infty$.

(a) Prove that $u(a) = \frac{1}{\pi r^2} \int \int_{B_r(a)} u(x + iy) \, dx \, dy$, for every $a \in \mathbb{C}$ and $r > 0$. Here, $B_r(a) = \{ z \in \mathbb{C} | |z - a| < r \}$ denotes the open ball of radius $r$ centered at $a$.

(b) Prove that $u(z) = 0$, for every $z \in \mathbb{C}$. 