Problem 1 (60 points). Determine if each of the following statements is true or false. If you decide that it is true, then please give a brief proof. If you decide that it is false, then please give a counterexample or prove your assertion. For your proof, you may cite a proved result from the text or lectured in the class, with a brief explanation how the conclusion follows.

(1) Let \((X, \mathcal{M}, \mu)\) be a measure space and \(E_j \in \mathcal{M} \ (j = 1, 2, \ldots)\). Denote by \(E = \{x \in X : x \in E_j \text{ for infinitely many } j\}\). If \(\sum_{j=1}^{\infty} \mu(E_j) < \infty\), then \(\mu(E) = 0\).
(2) Let $f \in L^1_{\text{loc}}(\mathbb{R}^n)$ and denote the Lebesgue set of $f$ by $L_f$. Let $x_0 \in \mathbb{R}^n$ and suppose $f$ is continuous at $x_0$. Then $x_0 \in L_f$.

(3) Let $H$ be a real Hilbert space. Let $x_k \in H$ ($k = 1, 2, \ldots$) and $x \in H$. If $x_k \to x$ weakly in $H$ and $\|x_k\| \to \|x\|$, then $\|x_k - x\| \to 0$. 
(4) Let $S$ denote the Schwartz space on $\mathbb{R}^n$. Let $f, g \in S$. If $f \ast g = 0$ in $\mathbb{R}^n$ then either $f = 0$ in $\mathbb{R}^n$ or $g = 0$ in $\mathbb{R}^n$.

(5) Let $f(x) = |x|$ ($x \in \mathbb{R}$) and identify $f$ as a distribution on $\mathbb{R}$. Then the second-order distributional derivative $f''$ is the 0 distribution (as the 0 linear functional) on $\mathcal{D}(\mathbb{R})$. 
Problem 2 (15 points). Let $\mu$ be the Lebesgue–Stieltjes measure associated to the following increasing and right-continuous function $F : \mathbb{R} \to \mathbb{R}$:

$$F(x) = \begin{cases} 
0 & \text{if } x < 0, \\
x + 2 & \text{if } 0 \leq x < 1, \\
4x^2 & \text{if } 1 \leq x < \infty.
\end{cases}$$

Calculate $\mu((0, 0])$, $\mu(\{1\})$, and $\mu([1, 2])$. 
Problem 3 (10 points). Let $X$ be a compact Hausdorff topological space. Let $K_j$ ($j = 1, 2, \ldots$) be a sequence of decreasing, nonempty compact subsets of $X$. Prove that $\cap_{j=1}^{\infty} K_j \neq \emptyset$. 
Problem 4 (25 points). Let $0 = x_0^{(k)} < x_1^{(k)} < \cdots x_{k-1}^{(k)} < x_k^{(k)} = 1$ ($k = 1, 2, \ldots$) and $0 \neq A_j^{(k)} \in \mathbb{R}$ ($j = 0, \ldots, k; \ k = 1, 2, \ldots$). Define for any $f \in C([0, 1])$

$$I[f] = \int_0^1 f(x) \, dx \quad \text{and} \quad I_k[f] = \sum_{j=0}^k A_j^{(k)} f\left( x_j^{(k)} \right) \quad (k = 1, 2, \ldots).$$

1. Assume $\sup_{k \geq 1} \sum_{j=0}^k \left| A_j^{(k)} \right| < \infty$ and $\lim_{k \to \infty} I_k[p] = I[p]$ for any polynomial $p$. Prove $\lim_{k \to \infty} I_k[f] = I[f]$ for $f \in C([0, 1])$.

2. (a) For any $k \geq 1$, $I_k$ is a linear functional on the Banach space $C([0, 1])$ (with the uniform norm). Prove that $\|I_k\| = \sum_{j=0}^k \left| A_j^{(k)} \right|$.

(b) Assume $\lim_{k \to \infty} I_k[f] = I[f]$ for any $f \in C([0, 1])$. Prove $\sup_{k \geq 1} \sum_{j=0}^k \left| A_j^{(k)} \right| < \infty$. 


Problem 5 (20 points). Let $X$ be a Banach space and denote by $\mathcal{L}(X)$ the space of all linear and bounded operators from $X$ to $X$. Let $T \in \mathcal{L}(X)$ be topologically isomorphic, i.e., $T : X \to X$ is linear, bijective, and both $T$ and $T^{-1}$ are continuous. Let $S \in \mathcal{L}(X)$ be such that $\|(S - T)T^{-1}\| < 1$. Prove that $S : X \to X$ is also topologically isomorphic.
Problem 6 (25 points). Let $(X, \mathcal{M}, \mu)$ be a measure space with $\mu(X) < \infty$. Let $1 < p < \infty$. Suppose $f_k \in L^p(\mu) \ (k = 1, 2, \ldots)$ are such that $\sup_{k \geq 1} \|f_k\|_{L^p(\mu)} < \infty$ and $f_k \to f$ in $L^1(\mu)$ for some $f \in L^1(\mu)$. Prove that $f \in L^p(\mu)$ and $f_k \to f$ in $L^q(\mu)$ for any $q \in (1, p)$. 
Problem 7 (25 points). Let $X$ be a locally compact Hausdorff topological vector space. Let $f \in C_0(X)$ and $f_k \in C_0(X)$ ($k = 1, 2, \ldots$). Prove that $f_k \to f$ weakly in $C_0(X)$ if and only if $\sup_{k \geq 1} \|f_k\|_u < \infty$ and $f_k \to f$ pointwise on $X$. 
Problem 8 (20 points). Let \( f \in L^1(\mathbb{R}) \) and \( xf(x) \in L^1(\mathbb{R}) \). Prove that the Fourier transform \( \hat{f} \) is differentiable at every point \( \xi \in \mathbb{R} \), and

\[
\frac{d}{d\xi} \hat{f}(\xi) = (-ixf)(\xi) \quad \forall \xi \in \mathbb{R}.
\]