Complex Analysis Qualifying Exam – Spring 2017

Name: ________________________________

Student ID: ________________________________

Instructions:
You have 3 hours. No textbooks and notes are allowed. Solve 7 of the following 8 questions.

Notation: \( \mathbb{D} = \{ z \in \mathbb{C} \mid |z| < 1 \} \).

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Problem 1. [10 points.]

Using the calculus of residues, compute

$$\int_0^\infty \frac{\cos x}{(1+x^2)^\frac{3}{2}} \, dx.$$
Problem 2. [10 points.]

Let $f : \mathbb{C} \to \mathbb{C}$ be entire. Assume that the function $g(z) = f(z) \cdot f\left(\frac{1}{z}\right)$ is bounded on $\mathbb{C} \setminus \{0\}$. Show that $f(z) = cz^m$. 

Problem 3. [10 points.]

Let \( n \geq 0 \) be an integer. Assume \( f : \mathbb{C} \to \mathbb{C} \) is an entire function such that

\[
|f(z)| \leq 1 + (\log(1 + |z|))^n.
\]

Prove that \( f \) is constant.
Problem 4. [10 points; 5, 5.]

(i) Construct an entire function with simple zeros at \( \left\{ \sqrt{n} + \frac{1}{\sqrt{n}} : n = 1, 2, \ldots \right\} \) and no other zeroes.

(ii) Construct a meromorphic function with simple poles at \( z = n\sqrt{n} \) and residues equal to \( \sqrt{n} \) for \( n = 1, 2, \ldots \).
Problem 5. [10 points.]

Let \( \mathcal{F} \) be the family of holomorphic functions \( f : \mathbb{D} \to \mathbb{C} \) defined over the open unit disc with

\[
f(0) = 1 \text{ and } \Re f > 0.
\]

Show that \( \mathcal{F} \) is a normal family.
Problem 6. [10 points.]

Find the number of roots of the polynomial

\[ z^{87} + 36z^{57} + 71z^4 + z^3 - z + 1 \]

in the region \( 1 \leq |z| \leq 2. \)
Problem 7. [10 points.]

Show that the function $f(z) = \cos(\sqrt{z})$ is entire. Determine the order, rank and genus of $f$. 
Problem 8. [10 points; 5, 5.]

(i) Let $u : \mathbb{C} \to \mathbb{R}$ be a harmonic function which is bounded. Show that $u$ is constant.

(ii) Let $H = \{ z : \ \text{Im } z > 0 \}$ denote the upper half plane. Let $u : \overline{H} \to \mathbb{R}$ be a continuous bounded function which is harmonic in $H$ and $u = 0$ on $\partial H$. Show that $u$ is constant.