Ph.D./Masters Qualifying Examination in Numerical Analysis

Examiner: Michael Holst

9am–12pm
Friday May 26, 2017

Name

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- Put your name in the box provided and staple exam to your solutions.
- Write your name clearly on every sheet of paper you submit.
1 Numerical Linear Algebra (270A)

Question 1.1. Let $A \in \mathbb{R}^{n \times n}$, and let $\| \cdot \|_p$ denote the standard $l^p$ norms on $\mathbb{R}^n$, $1 \leq p \leq \infty$.

(a) Show the following norm equivalence relations for the $l^p$-norms on $\mathbb{R}^n$:
\[
\|u\|_\infty \leq \|u\|_2 \leq \|u\|_1 \leq \sqrt{n}\|u\|_2 \leq n\|u\|_\infty, \quad \forall u \in \mathbb{R}^n,
\]
and then show the following induced matrix norm and spectral radius relationships:
\[
\|A\|_1 \leq \sqrt{n}\|A\|_2 \leq n\|A\|_1, \quad \|A\|_\infty \leq \sqrt{n}\|A\|_2 \leq n\|A\|_\infty, \quad \rho(A) \leq \|A\|_p.
\]

(b) Assume $A$ is invertible and use the results from (a) to derive analogous relationships for $\kappa_p(A)$.

(c) Assume $A$ is invertible, and $Ax = b$ and $A(x + \delta x) = (b + \delta b)$ for some $x, b, \delta x, \delta b \in \mathbb{R}^n$. Show
\[
\frac{\|\delta x\|_p}{\|x\|_p} \leq \kappa_p(A) \frac{\|\delta b\|_p}{\|b\|_p}, \quad \frac{\|\delta b\|_p}{\|b\|_p} \leq \kappa_p(A) \frac{\|\delta x\|_p}{\|x\|_p}, \quad 1 \leq p \leq \infty.
\]

Question 1.2. Let $A \in \mathbb{R}^{m \times n}$, $m \geq n$, and consider the overdetermined system:
\[
Ax = b, \quad \text{where } x \in \mathbb{R}^n, b \in \mathbb{R}^m.
\]

(a) Formulate the minimization problem that defines the least-squares solution, and derive the normal equations from this problem.

(b) Show that $A^T A$ is nonsingular if and only if $A$ has full rank.

(c) Identify the projector $P$ arising in least-squares, and show how to exploit a QR factorization.

Question 1.3. Let $A, B \in \mathbb{R}^{n \times n}$ be SPD matrices.

(a) Show that $A$ defines an inner-product and norm
\[
(u, v)_A = (Au, v)_2, \quad \|u\|_A = (u, u)_A^{1/2},
\]
where $(u, v)_2$ is the usual Euclidean 2-inner-product.

(b) Starting with the Caley-Hamilton Theorem, derive the Conjugate Gradient method for solving the preconditioned linear system: $BAu = Bf$. Mathematically justify each step of the derivation. The derivation will be based around building up an expanding set of Krylov subspaces, exploiting a 3-term recursion for generating an $A$-orthogonal bases for these subspaces, and enforcing minimization of the $A$-norm of the error at each iteration of the method.
2 Numerical Approximation and Nonlinear Equations (270B)

Question 2.1. Let $F : D \subset \mathbb{R}^n \to \mathbb{R}^n$ be continuously differentiable on an open convex set $D$.

(a) Derive the following expansion with integral remainder:

$$F(x + h) = F(x) + F'(x)h + \int_0^1 \{F'(x + \xi h) - F'(x)\} h \, d\xi,$$

and then use this expansion to derive Newton’s method for $F(x) = 0$.

(b) Assume that $F(x^*) = 0$ for some $x^* \in D$, and that $F'(x^*)$ is nonsingular. Prove the basic convergence theorem for Newton’s method: There exists an open neighborhood $S \subset D$ containing $x^*$ such that, for any $x_0 \in S$, the Newton iterates are well-defined, remain in $S$, and converge to $x^*$ at $q$-superlinear rate.

(c) Show that if the Jacobian $F'(x)$ is Lipschitz in the set $S$ in part (b) for some uniform Lipschitz constant, then the convergence rate is $q$-quadratic.

Question 2.2. Let $f : \mathbb{R}^n \to \mathbb{R}$, $c : \mathbb{R}^n \to \mathbb{R}^m$, $0 < m < n$, and consider the problem:

$$\min_{x \in \mathbb{R}^n} f(x),$$

subject to $c(x) = 0$.

Using some basic ideas from linear algebra, prove the main result that leads to the method of Lagrange Multipliers for this problem: If $f$ and $c$ are differentiable at a feasible point $x^*$, then

$$\nabla f(x^*)^T p \geq 0, \quad \forall p \text{ such that } c'(x^*)p = 0,$$

if and only if there exists a vector $\lambda^* \in \mathbb{R}^m$ such that $\nabla f(x^*) = c'(x^*)^T \lambda^*$.

Question 2.3. Consider the following tabulated information about a function $f : \mathbb{R} \to \mathbb{R}$:

<table>
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<tr>
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<tr>
<td>0</td>
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(a) Construct the (unique) quadratic interpolation polynomial $p_2(x)$ which interpolates the data.

(b) If the function $f(x)$ that generated the above data was actually the cubic polynomial $P_3(x) = 2x^3 - 2x^2 + 1$, derive an error bound (a fairly “tight” one) for the interval $[0, 2]$.

(c) Use the composite trapezoid rule with two intervals to construct an approximation to:

$$\int_0^2 f(x) \, dx,$$

and give an expression for the error.
3 Numerical Ordinary Differential Equations (270C)

**Question 3.1.** We consider now the problem of best $L^p$-approximation of a function $u(x) = x^4$ over the interval $[0, 1]$ from a subspace $V \subset L^p([0, 1])$.

(a) Determine the best $L^2$-approximation in the subspace of linear functions; i.e., $V = \text{span}\{1, x\}$, and justify the technique you use.

(b) Precisely formulate the best approximation problem in the case $p \neq 2$, and propose an algorithm for finding the solution.

(c) Let $X$ be a general Hilbert space, and let $U \subset X$ be a subspace. Prove that the orthogonal projection of $u$ onto $Pu \in U$ is the best approximation, and that this projection is unique.

**Question 3.2.** Consider the initial value problem in ordinary differential equations:

$$y' = f(t, y), \quad t \in (a, b)$$

$$y(a) = \alpha.$$

(a) Derive the Taylor method of order 2 using Taylor expansion of the solution to the ODE.

(b) Derive the Runge-Kutta method of order 2 (by matching terms in the Taylor method).

(c) Consider the multistep method (3-step Adams-Bashford):

$$w_0 = \alpha, \quad w_1 = \alpha_1, \quad w_2 = \alpha_2,$$

$$w_{i+1} = w_i + \frac{h}{12} [23f(t_i, w_i) - 16f(t_{i-1}, w_{i-1}) + 5f(t_{i-2}, w_{i-2})], \quad i = 2, 3, \ldots, N - 1.$$

Determine the local truncation error, and examine the stability using the root condition. Finally, draw a conclusion about the convergence properties of the method.