Name: _____________________________________________

Student ID: _________________________________________

Instructions:

You have 3 hours. No textbooks and notes are allowed. Solve 7 of the following 8 questions.

Notation: \( \mathbb{D} = \{ z \in \mathbb{C} \mid |z| < 1 \} \).

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**Problem 1.** [10 points.]

Using the calculus of residues, compute

\[ \int_{0}^{\infty} \frac{\log^2 x}{1 + x^2} \, dx. \]

You may wish to use the contour consisting of two half circles in the upper half plane, one of small radius and another of large radius, connected by two line segments along the real axis.
**Problem 2.** [10 points.]

Let $U \subset \mathbb{C}$ be an open set that contains the closed unit disc $\overline{D} \subset U$. Show that for all holomorphic functions $f : U \to \mathbb{C}$ we have

$$\max_{|z|=1} \left| f(z) - \frac{e^z}{z} \right| \geq 1.$$
Problem 3. [10 points.]

Find a bijective analytic function \( f \) from \( \{ z : |z| < 1, \Re z > 0 \} \) to \( \{ z : \Im z > 0 \} \).
Problem 4. [10 points.]

Let \( h^+ = \{ z : \text{Im } z > 0 \} \) denote the upper-half plane. Let \( \mathcal{F} \) be the family of holomorphic functions \( f : h^+ \to \mathbb{C} \) such that

\[
f(i) = 0 \text{ and } |f(z)| < 1 \text{ for all } z \in h^+.
\]

Find the maximum value of \( |f(2i)| \) for \( f \in \mathcal{F} \).
Problem 5. [10 points.]

Prove that the Riemann surface of the complete analytic function associated to a branch of log \( z \) is simply connected.
Problem 6. [10 points.]

(a) Prove that for any distinct real numbers $r$ and $\rho$,

$$\frac{1}{2\pi} \int_0^{2\pi} \log |re^{it} - \rho| \, dt = \max(\log r, \log |\rho|).$$

(b) Show that the series

$$u(z) = \sum_{n=0}^{\infty} \frac{1}{2^n} \log \left| z - \frac{1}{2^n} \right|$$

defines a subharmonic function.
Problem 7. [10 points.]

Assume that $f$ and $g$ are entire functions without common zeros. Show that there exist entire functions $A, B$ such that

$$Af + Bg = 1.$$ 

Hint: Ensure that $A = (1 - Bg)/f$ is entire by matching the principal parts of $1/f$ and $Bg/f$. 
Problem 8. [10 points.]

Let $n \geq 1$ be an integer. Let $u : \mathbb{C} \to \mathbb{R}$ be a harmonic function such that

$$|u(x)| \leq C(1 + |x|^n)$$

for all $x \in \mathbb{C}$.

Show that $u$ is a polynomial.