1. Let $S_3$ be the symmetric group on 3 letters. If we pick elements $\tau, \sigma \in S_3$ of orders 2 and 3 respectively, we get a surjective homomorphism $\theta : \mathbb{Z}_2 \ast \mathbb{Z}_3 \to S_3$. By constructing a suitable covering space of a 2-complex, show that the kernel of $\theta$ is a free group of rank 2.

2. Let $S^2$ be the standard unit sphere, and let $R_\theta : S^2 \to S^2$ be the operation of rotation through angle $\theta$ anticlockwise about the $z$-axis. Let $M$ be the closed 4-manifold obtained by gluing together two copies $A_1, A_2$ of $B^2 \times S^2$ along their common boundary $S^1 \times S^2$; specifically, identify $(e^{i\theta}, v) \in \partial A_1 \sim (e^{i\theta}, R_\theta(v)) \in \partial A_2$ for all $e^{i\theta} \in S^1, v \in S^2$.

Use Mayer-Vietoris to compute $H_*(M; \mathbb{Z})$. Give an example of another closed 4-manifold $N$ with the same homology, and use intersection theory to show that $M$ and $N$ are not homotopy-equivalent.

3. Let $X_n$ be the space formed from the disjoint union of $n$ copies $C_1, \ldots, C_n$ of the cylinder $S^1 \times I$ by gluing, for each $k$, the $S^1 \times \{1\}$ of $C_k$ to the $S^1 \times \{0\}$ of $C_{k+1}$ using a map of degree $k$. There is a natural sequence of inclusions $X_1 \subseteq X_2 \subseteq X_3 \subseteq \cdots$ and so we may define $X$ to be the direct limit of this family. ($X$ is called a mapping telescope.) What is $H_1(X; \mathbb{Z})$? (You might find it helpful to view $X$ as a CW-complex, but you don’t have to.)

4. Let $Y$ be a space obtained by attaching a 4-ball, via a degree 6 map of its boundary, to a 3-sphere. Calculate the integral homology $H_*(Y \times \mathbb{R}P^2; \mathbb{Z})$.

5. Given any map $f : S^n \to S^n$, let $\Gamma_f = \{(x, f(x)) : x \in S^n\} \subseteq S^n \times S^n$ be the graph of $f$. By using the intersection theory of $S^n \times S^n$, calculate the intersection number $[\Gamma_{\text{id}}], [\Gamma_f]$ and deduce that $f$ must have at least one fixed point provided $\deg f \neq (-1)^{n+1}$.

6. Let $M^3$ be a homology sphere – a closed 3-manifold having the same homology groups as $S^3$ – and let $X = \Sigma M$ be its suspension. What are the fundamental group and homology groups of $X$? Show that $X$ is homotopy-equivalent to $S^4$.

7. Suppose that $S^3 = M \cup_{\Sigma} N$ is a decomposition of the 3-sphere into two compact 3-manifolds, glued along their common boundary surface $\Sigma$. Prove that $H^1(N) \cong H_1(M)$, and conclude that $\mathbb{R}P^2$ cannot be embedded in $S^3$. 

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