Problem 1.

A. Show that when $G$ is transitive over $\Omega$, the risk function of any equivariant estimator is constant.

B. Suppose $\rho$ is convex and even function and that $Z$ is a random variable with a symmetric distribution about 0. Show that $\phi : a \mapsto \mathbb{E}[\rho(Z - a)]$ is convex and even.

C. In a location-scale setting, show that a function $q$ is invariant if and only if $q(x_1, \ldots, x_n) = u(z_1, \ldots, z_{n-2})$ where $z_j = (x_j - x_n)/(x_{n-1} - x_n)$. Deduce that an estimator $\delta$ is equivariant in this setting if and only if $\delta(x_1, \ldots, x_n) = \delta_0(x_1, \ldots, x_n) - u(z_1, \ldots, z_{n-2})$, where $\delta_0$ is a given equivariant statistic and $u$ is arbitrary. Give an example of $\delta_0$. 
D. Prove M1 from the summary sheet.

E. Prove M2 from the summary sheet.

F. Consider testing $X \sim f_0$ versus $X \sim f_1$, where $f_0$ and $f_1$ are densities with respect to some measure. Assume that $f_1(X)/f_0(X)$ has a continuous distribution when $X \sim f_0$. Give a most powerful test at level $\alpha$. Is it unique?

G. Define what it means for a test to be unbiased level $\alpha$ for testing $\theta \in \Omega_H$ versus $\theta \in \Omega_K$. 
Problem 2. Consider $X_1, \ldots, X_n$ iid uniform in $[a - b/2, a + b/2]$, where $a \in \mathbb{R}$ and $b > 0$ are both unknown.

A. Show that this is a location-scale family.

B. Show that $X_{(1)} = \min_i X_i$ and $X_{(n)} = \max_i X_i$ are jointly sufficient.

C. Derive the MLE for $a$. (If it is not unique, make a natural choice if possible.) Is it equivariant in some way?

D. With square error loss, derive the MRE for $a$. 
Problem 3. Let $X_1, \ldots, X_n$ be iid $\mathcal{N}(a, \sigma^2)$, and independently, let $Y_1, \ldots, Y_n$ be iid $\mathcal{N}(b, \sigma^2)$.

A. Assume $\sigma^2$ is known. Consider testing $a \geq b$ versus $a < b$. Derive a UMP test at level $\alpha$. [There is a point mass prior that is least favorable.]

B. Assume $\sigma^2$ is known. Consider testing $a = b$ versus $a \neq b$. Derive a UMPU test at level $\alpha$.

C. Assume $\sigma^2$ is unknown. Consider testing $a = b$ versus $a \neq b$. Is there a UMPU test at level $\alpha$?