QUALIFYING EXAMS

Spring 2020

Three-hour exam. Do as many questions as you can. No book or notes allowed. Each is worth 4 marks. Please write clear maths and clear English which could be understood by one of your fellow students! Include as much detail as is appropriate; you can use standard results and theorems in your answers provided you refer to them clearly. Even if you can not solve the whole problem, you still need to write your partial answer to receive partial credit.

1. Consider the group $G = \mathbb{Z} \ast (\mathbb{Z} \oplus \mathbb{Z})$ (here $\ast$ means the free product). Realize $G$ as the fundamental group of some topological space. Then classify (up to isomorphism) all subgroups of $G$ of index two.

2. Let $n$ be a positive even number. Show that there does not exist a continuous map $f: S^n \to S^n$ such that for any $\vec{x} \in S^n$, we have $f(\vec{x}) \neq \vec{x}$ and $f(\vec{x}) \neq -\vec{x}$. (Here $-\vec{x}$ denotes the antipodal point of $\vec{x}$.)

3. Let $X$ be a 3-dimensional CW complex obtained by attaching a single 3-dimensional cell to $S^2$ via an attaching map of degree 3. Compute the homology group $H_k(X \times X; \mathbb{Z})$ for all $k \geq 0$.

4. Show that $\mathbb{C}P^2$ has no orientation reversing self-homeomorphism.

5. Let $M$ be a closed, orientable, connected manifold of dimension $n$. Suppose there is a continuous map $f: S^n \to M$ with nonzero mapping degree. Show that $H_k(M; \mathbb{Q}) = H_k(S^n; \mathbb{Q})$ for any $k \geq 0$. (Hint: use the Poincaré duality.)

6. Consider a continuous map $f: \mathbb{R}P^n \to \mathbb{R}P^n$, where $n$ is a positive even number. Show that $f$ has a fixed point. (Here $\mathbb{R}P^n$ is the $n$-dimensional real projective space obtained by identifying antipodal points of the sphere $S^n$.)

7. Compute the homotopy group $\pi_3(T^2 \vee S^3)$. Here $\vee$ means one-point union of two topological spaces (wedge) and $T^2$ denotes the 2 dimensional torus.

8. Let $X$ be an $n$-dimensional closed smooth manifold. Suppose $X$ is simply connected and suppose

$$H_k(X; \mathbb{Z}) = H_k(S^n; \mathbb{Z})$$

for all $k \geq 0$. Show that $X$ is homotopy equivalent to $S^n$. 

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