Stochastic Realization of $\sigma$-Algebras

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Abstract

Stochastic realization problems are motivated by control and signal processing for stochastic systems. A stochastic system is defined by the property that at each time the current state makes the future and the past of the output and state processes conditionally independent. The stochastic realization problem for a $\sigma$-algebra family is then to construct the state $\sigma$-algebra family such that the families jointly satisfy the definition of a stochastic system. The first result is a characterization of minimal $\sigma$-algebra which make two $\sigma$-algebras minimally conditionally independent. The second result allows the construction of state $\sigma$-algebra families which together with a considered output $\sigma$-algebra family forms a stochastic system.

1 Introduction

Results are presented for the stochastic realization problem in the setting of $\sigma$-algebras.

Problems of control and signal processing require theoretical models in the form of deterministic or of stochastic systems. Stochastic systems are described in discrete time by mappings from the state and the input to the probability distributions of the next state and the current output and in continuous time by stochastic differential equations. For the filtering problem the Kalman filter or generalizations of it are applied. For stochastic control problems there are control laws for minimum variance cost functions and for cost functions of quadratic forms in the state and the input.

As has first been pointed out by R.E. Kalman, control and filtering theory is much advanced by the restriction for those problems to those stochastic systems for which finite-dimensional filters or finite-dimensional controllers exist. At Kalman’s insistence, researchers have started research on the stochastic realization problem. The weak stochastic realization problem for Gaussian processes was solved by P. Faurre, the strong stochastic realization problem for Gaussian processes by A. Lindquist, G. Picci, and G. Ruckebusch, see [2] and the references provided in that paper. See also [3] for the importance of stochastic realization for system identification.

The author of this paper has formulated the stochastic realization problem for $\sigma$-algebras and for $\sigma$-algebraic families. For publications see, [6, 8, 9, 10, 11]. The realization problem for $\sigma$-algebras was partly solved in [10]. The characterization of minimal realizations is not described in that paper. Others have carried out related research, see [4]. The solution to a special case of the stochastic realization problem for $\sigma$-algebra families was solved by W.J. Stronegger, see [5]. The interest of the stochastic realization problem for $\sigma$-algebra families is in the development of general theory for stochastic control and filtering, not only for Gaussian processes but also for finite-valued processes. Stochastic realization theory for $\sigma$-algebras is inspired by that for Hilbert spaces yet more general.

The first result of this paper is a new characterization of minimal realizations for the $\sigma$-algebraic stochastic realization problem. The solution is inspired by the corresponding problem for Hilbert spaces. The second result is a characterization of stochastic realizations of $\sigma$-algebra families. Procedures for the construction of realizations have been partly developed.

The outline of the paper follows. Preliminaries on probability theory are mentioned in Section 2. Section 3 provides results on the stochastic realization problem for $\sigma$-algebras. Section 4 provides results on the stochastic realization for $\sigma$-algebra families.
In this section notation is introduced.

Let \((\Omega, F, P)\) denote a complete probability space consisting of a set \(\Omega\), a \(\sigma\)-algebra \(F\), and a probability measure \(P\). Let

\[
F = \left\{ G \subseteq F : \text{G is a } \sigma\text{-algebra containing all null sets of } F \right\}. \tag{1}
\]

If \(F_1, F_2 \in F\) then \(F_1 \lor F_2\) denotes the smallest \(\sigma\)-algebra in \(F\) that contains both \(F_1\) and \(F_2\). For \(G \in F\) let

\[
L^+(G) = \{ x : \Omega \to R_+ \mid x \text{ is } G \text{ measurable} \}.
\]

If \(x : \Omega \to R^n\) is a random variable then denote by \(F^x\) the \(\sigma\)-algebra generated by \(x\). Denote by \((F_1, F_2, \ldots, F_n) \in I\) that the indicated \(\sigma\)-algebras are independent.

**Definition 2.1** The conditional independence relation for a triple of \(\sigma\)-algebras is defined by the condition that

\[
E[x_1 | G] = E[x_1 | F_1] E[x_2 | G], \tag{2}
\]

for all \(x_1 \in L^+(F_1)\), \(x_2 \in L^+(F_2)\), and for \(F_1, F_2, G \in F\). Denote this condition by \((F_1, F_2, G) \in CI\) and one says that \(F_1, F_2\) are conditionally independent given \(G\) or that \(G\) splits \(F_1\) and \(F_2\).

It is an easily established fact that \((F_1, F_2, G) \in CI\) iff

\[
E[x_1 | F_2 \lor G] = E[x_1 | G], \tag{3}
\]

for all \(x_1 \in L^+(F_1)\), see [1, II.45]. If \(F_1 \subseteq G\) then \((F_1, F_2, G) \in CI\). Also, if \((F_1, F_2) \subseteq I\) then \((F_1, F_2, G) \in CI\). Consider on the set of \(\sigma\)-algebras the partial order of the inclusion relation. One says that \(F_1, F_2, G\) form a minimal triple if \((F_1, F_2, G) \in CI\) and \(G\) is minimal; specifically, if \((F_1, F_2, G) \in CI\) and for any \(\sigma\)-algebra \(G_1 \subseteq G\), \((F_1, F_2, G_1) \in CI\) implies that \(G_1 = G\). Then one writes that \((F_1, F_2, G) \in CI_{\min}\).

For \(F_1, F_2 \in F\) define

\[
\sigma(F_2 | F_1) = \sigma \left( \left\{ E[x_2 | F_1] : \Omega \to R_+ \mid \forall x_2 \in L^+(F_2) \right\} \right), \tag{4}
\]

where the right-hand side of this formula denotes the \(\sigma\)-algebra generated by all random variables in the indicated set.

### 3 Stochastic realization of a pair of \(\sigma\)-algebras

**Problem 3.1** Consider \(F_1, F_2 \in F\). Construct all \(\sigma\)-algebras \(G \in F\) such that:

\[
G \subseteq (F_1 \lor F_2), \quad (F_1, F_2, G) \in CI_{\min}.
\]

Define

\[
G(F_1, F_2) = \left\{ G \in F | \forall (F_1, F_2, G) \in CI_{\min} \right\}, \tag{5}
\]

\[
G_{\min}(F_1, F_2) = \left\{ G \in G(F_1, F_2) \mid (F_1, F_2, G) \in CI_{\min} \right\}. \tag{6}
\]

For \(F_1, F_2 \in F\) define

\[
F_{21} = \sigma(F_2 | F_1), \quad F_{12} = \sigma(F_1 | F_2). \tag{7}
\]

**Proposition 3.2** [8, 10]. Consider \(F_1, F_2, G \in F\). Assume that \((F_1, F_2, G) \in CI\) and \(G \subseteq F_1 \lor F_2\). Then \((F_1, F_2, G) \in CI\) if and only if \((F_{21}, F_{12}, G) \in CI\). Moreover, \((F_1, F_2, G) \in CI_{\min}\) if and only if \((F_{21}, F_{12}, G) \in CI_{\min}\).

There exist \(F_1, F_2, G \in F\) such that \((F_1, F_2, G) \in CI\) and \(G \neq F_1 \lor F_2\). There exist \(F_1, F_2, G \in F\) such that \((F_1, F_2, G) \in CI_{\min}\), \(G \neq F_1 \lor F_2\), and \(G \neq F_1 \lor F_1\). Below attention is restricted to \(G \subseteq F_1 \lor F_2\) and hence to \(G_{\min}(F_1, F_2)\). The characterization of minimal \(\sigma\)-algebras which make two given \(\sigma\)-algebras conditional independent is expressed best in terms of a pair of \(\sigma\)-algebras. The following approach is inspired by the analogous problem for Hilbert spaces. One may call the \(\sigma\)-algebras \(H_1\) and \(H_2\) the forward and the backward \(\sigma\)-algebras respectively.

**Definition 3.4** Consider \(F_1, F_2 \in F\). Define the set

\[
H = H(F_1, F_2) = \left\{ (H_1, H_2) : (H_1, H_2) \subseteq F_1 \times F_2, \quad \forall H_1 \subseteq H_1 \subseteq F_1 \lor F_2, \quad \forall H_2 \subseteq H_2 \subseteq F_1 \lor F_2, \quad H_1 \cap H_2 = \sigma(H_1 | H_2) = \sigma(H_2 | H_1) \right\}. \tag{8}
\]

Define on this set the partial order

\[
(H_{11}, H_{12}) \preceq (H_{21}, H_{22}), \tag{9}
\]

if \(H_{11} \subseteq H_{21}\) and if \(H_{12} \subseteq H_{22}\).
An element \((H_1, H_2) \in \mathbf{H}\) is said to be minimal if it is minimal with respect to this partial order. Define the set
\[
\mathbf{H}_{\min}(F_1, F_2) = \{(H_1, H_2) \in \mathbf{H}(F_1, F_2) | (H_1, H_2)\text{minimal}\}.
\]

Define the map
\[
f : \mathbf{H}(F_1, F_2) \to \mathbf{G}(F_1, F_2),
\]
\[
f(H_1, H_2) = H_1 \cap H_2.
\]

**Proposition 3.5** Consider \(F_1, F_2 \in \mathbf{F}\).

(a) The function \(f\) is well defined.

(b) The function \(f\) is a bijection and
\[
f^{-1}(G) = (F_{21} \lor G, F_{12} \lor G).
\]

There exists an example for which \(\mathbf{H}_{\min}(F_1, F_2) \subset \mathbf{H}(F_1, F_2)\).

**Theorem 3.6** Consider \(F_1, F_2 \in \mathbf{F}\). The restriction of the function \(f : \mathbf{G} \to \mathbf{H}\) to \(f : \mathbf{G}_{\min} \to \mathbf{H}_{\min}\) is a bijection.

Problem 3.1 is with Theorem 3.6 replaced by a problem to characterize the set \(\mathbf{H}_{\min}(F_{21}, F_{12})\). The construction of the elements of this set is partly solved and will be included in the final version of the paper. The problem for the case in which the \(\sigma\)-algebras \(F_1\) and \(F_2\) are generated by finite-dimensional Gaussian random variables is solved in [7].

4 Stochastic realization problem for a \(\sigma\)-algebra family

Consider the discrete-time index set of the integers, \(T = \mathbb{Z}\) and the \(\sigma\)-algebra families \(\{F_t, G_t, t \in T\}\). Define
\[
F_t^+ = \lor_{s \geq t} F_s, \quad F_t^- = \lor_{s \leq t} F_s,
\]
\[
G_t^+ = \lor_{s \geq t} G_s, \quad G_t^- = \lor_{s \leq t} G_s.
\]

**Definition 4.1** The \(\sigma\)-algebra families \(\{F_t, G_t, t \in T\}\) are said to form a \(\sigma\)-algebraic system if for all \(t \in T\)
\[
(F_t^+ \lor G_t^+, F_{t-1}^- \lor G_t^- | G_t) \in CI.
\]

The reader may think of a \(\sigma\)-algebraic system as related to a discrete-time stochastic system with a state and output process. Thus, \(F_t = F_{t-1}^+\) is the \(\sigma\)-algebra generated by the output at time \(t \in T\) and \(G_t = F_{t}^-\) is the \(\sigma\)-algebra generated by the state at time \(t\). Condition (13) translates in words to: At any time \(t \in T\) the current state \(\sigma\)-algebra \(G_t\) makes the future of the output and of the state conditional independent of the past of the output and of the state. The asymmetry of future and past with respect to the \(\sigma\)-algebra family \(\{F_t, t \in T\}\) is necessary because of technical reasons.

**Problem 4.2** Consider the \(\sigma\)-algebra family \(\{F_t, t \in T\}\).

(a) Does there exist a \(\sigma\)-algebra family \(\{G_t, t \in T\}\) on the same probability space of \(\{F_t, t \in T\}\) such that \(\{F_t, G_t, t \in T\}\) form a \(\sigma\)-algebraic system. Call \(\{F_t, G_t, t \in T\}\) a realization of the family \(\{F_t, t \in T\}\).

(b) Call \(\{F_t, G_t, t \in T\}\) a minimal realization of \(\{F_t, t \in T\}\) if (1) it is a realization and if (2) for any other realization \(\{F_t, \tilde{G}_t, t \in T\}\), \(\tilde{G}_t \subseteq G_t\) for all \(t \in T\) implies \(\tilde{G}_t = G_t\). Characterize a minimal realization.

(c) Classify all \(\sigma\)-algebra families \(\{G_t, t \in T\}\) such that (1) \(G_t \subseteq F_{\infty}^-\) for all \(t \in T\) and

(2) \(\{F_t, G_t, t \in T\}\) is a minimal realization of \(\{F_t, t \in T\}\).

The solution procedure for the above problem makes use of the result of the realization problem of \(\sigma\)-algebras as discussed in Section 3. Consider the \(\sigma\)-algebra family \(\{F_t, t \in T\}\) and for all \(t \in T\) let \(G_t \subseteq F_{\infty}^-\) be such that
\[
(F_t^+, F_{t-1}^+ | G_t) \in CI.
\]

In general the \(\sigma\)-algebra family \(\{F_t, G_t, t \in T\}\) does not form a \(\sigma\)-algebraic system. A condition is needed.

**Definition 4.3** A \(\sigma\)-algebra family \(\{F_t, G_t, t \in T\}\) is called transitive if

1. \(\{F_{t-1}^+ \lor G_t, t \in T\}\) is increasing in \(T\), or, equivalently, if
   \[s < t \Rightarrow (F_{s-1}^+ \lor G_s) \subseteq (F_{t-1}^+ \lor G_t);\]

2. \(\{F_t^+ \lor G_t, t \in T\}\) is decreasing in \(T\), or, equivalently, if
   \[s < t \Rightarrow (F_s^+ \lor G_s) \supseteq (F_t^+ \lor G_t).\]
Proposition 4.4 Consider the $\sigma$-algebra families 
$\{F_t, G_t, t \in T\}$. Assume that 

$$(F_t^+, F_{t-1}^- | G_t) \in CI, \forall t \in T.$$ 

Then 

$$(F_t^+ \vee G_t^+ \vee F_{t-1}^- \vee G_t^- | G_t) \in CI, \forall t \in T,$$ 

if and only if $\{F_t, G_t, t \in T\}$ is transitive.

Below $\sigma$-algebra families $\{H_t^+, H_t^-; t \in T\}$ are considered that may be interpreted as forward and backward families respectively. These concepts are inspired by the analogous Hilbert space case, see [2].

Theorem 4.5 Consider $\sigma$-algebra families 

$\{F_t, H_t^-, H_t^+, t \in T\}$. 

Define the conditions:

1. $\{H_t^+, t \in T\}$ is an increasing family and $\{H_t^-, t \in T\}$ is a decreasing family.

2. For all $t \in T$, $H_t^+ \vee H_t^- = F_t^+ \vee F_t^-$. 

3. For all $t \in T$, $F_t^+ \subseteq H_t^+$ and $F_t^- \subseteq H_t^-$. 

4. For all $t \in T$, 

$$(H_t^+, H_t^- | H_t^+ \cap H_t^-) \in CI.$$ 

(a) Assume that the $\sigma$-algebra families 

$\{F_t, G_t, t \in T\}$ 

form a $\sigma$-algebraic system such that for all $t \in T$, $G_t \subseteq F_{t-\infty}^-$. Define $H_t^+ = F_{t-1}^- \vee G_t$ and $H_t^- = F_t^+ \vee G_t$. Then the $\sigma$-algebra families $\{H_t^+, H_t^-, t \in T\}$ satisfy the conditions (1-4) above.

(b) Consider $\sigma$-algebra families 

$\{F_t, H_t^+, H_t^-, t \in T\}$ 

that satisfy the conditions (1-4) above. Define for all $t \in T$, $G_t = H_t^+ \cap H_t^-$. Then $\{F_t, G_t, t \in T\}$ form a $\sigma$-algebraic system.

With this theorem one can prove that the filter $\sigma$-algebraic system studied in [5] is a $\sigma$-algebraic stochastic realization.

5 Conclusions

The stochastic realization problem for $\sigma$-algebra families has been considered. A characterization has been presented of those $\sigma$-algebras which make two considered $\sigma$-algebras minimally conditionally independent. A second result allows the construction of state $\sigma$-algebra family which together with the output $\sigma$-algebra family form a stochastic system. Further research is required on the algorithms for the set $G_{min}(F_1, F_2)$ and for the properties of stochastic realizations.

References