Stability of linear delay systems. A note on frequency domain interpretation of Lyapunov-Krasovskii method

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Abstract

In this note, we study the problem of asymptotic stability of the delay linear systems. More precisely, we examine the conservativity of a widely-used approach of investigation, based on quadratic Lyapunov-Krasovskii functionals. It is shown that the success of this approach implies strong delay-independent stability of the system under study, but is in general a still stronger condition. Also, a necessary and sufficient condition is given, under which the Lyapunov-Krasovskii method is not conservative.

1 Introduction

We consider in this note the problem of asymptotic stability of the following delay system:

\[ \dot{x} = Ax + A_dx(t-h), \] (1)

where \( A, A_d \in \mathbb{R}^{n \times n} \). As is well-known, defining

\[ \mathcal{P}(s, z) \overset{\text{def}}{=} \det(sI - A - z A_d), \]

for any given \( h \geq 0 \), the asymptotic stability of (1) is equivalent to

\[ \forall s \in \mathbb{C}, \quad \text{Re} \, s \geq 0 \Rightarrow \mathcal{P}(s, e^{-sh}) \neq 0. \]

Both for reasons of computational simplicity and stability robustness, the issue of stability of (1) independent of the delay \( h \) has been considered, see [8, 9, 10] and the surveys [11, 14]. This property is expressed as:

\[ \forall h \geq 0, \forall s \in \mathbb{C}, \quad \text{Re} \, s \geq 0 \Rightarrow \mathcal{P}(s, e^{-sh}) \neq 0. \]

Hale et al. have provided necessary and sufficient condition of asymptotic stability independent of the delay [5, Theorem 2.4], namely:

\[ \text{Re} \, \lambda(A + A_d) < 0, \] (2a)

\[ \forall (s, z) \in \mathbb{C} \times \mathbb{R}, \quad s \neq 0, \quad |z| = 1 \Rightarrow \mathcal{P}(s, z) \neq 0, \] (2b)

where \( \lambda(A) \) designates the set consisting of the eigenvalues of \( A \), and \( \text{Re} \, \lambda(A) \) the set of their real parts (in the sequel, \( |\lambda(A)| \) denotes analogously the set of their moduli).

On the other hand, sufficient condition for asymptotic stability independent of the delay has been given, see [6, 12–14], using Lyapunov-Krasovskii functional approach. More precisely, for \( P, Q \in \mathbb{R}^{n \times n} \), positive definite, defining

\[ V(\phi) \overset{\text{def}}{=} \phi^T(0)P\phi(0) + \int_{-h}^0 \phi^T(s)Q\phi(s) \, ds \]

for \( \phi \in C([-h, 0]) \), the quadratic functional \( V(\phi) \) is bounded from below by \( \|\phi(0)\|^2 / \|P^{-1}\| \), and one has

\[ \forall t \in \mathbb{R}^+, \quad \frac{d}{dt}[V(x_t)] \leq \left( x(t) \right)^T R \left( x(t-h) \right), \]

\[ R \overset{\text{def}}{=} \begin{pmatrix} A^T P + PA + Q & PA_d \\ A_d^T P & -Q \end{pmatrix} \] (3)

along the trajectories of system (1) (where as usual \( x_t(s) = x(t + s), s \in [-h, 0] \)). Asymptotic stability of system (1) is then deduced when \( R < 0 \), using Krasovskii’s time-domain technique [12].

In the present paper, we attempt to clarify and compare the two previous stability criteria for delay systems. Our contribution consists in proving that the success of the Lyapunov-Krasovskii functional approach (that is the existence of positive matrices \( P, Q \) providing negative \( R \) in (3)) implies a certain frequency domain condition closely related to (2), and called strongly delay-independent stability by Niculescu et al. [13]. The proof of this fact involves the combined use of continuous-time and discrete-time versions of the Kalman-Yakubovich-Popov lemma\(^1\). Further results on the same subject are given in [3]. Results of absolute stability for nonlinear delay systems, obtained with the same mathematical technique, are given in [1, 2, 4].

2 Statement and proof of the result

Our main result is as follows.

\(^1\)The continuous-time version is due to Kalman [7] and Yakubovich [21], the discrete-time to Szegö and Kalman [16].
Theorem 1. (Strong [13] delay-independent stability criterion) Consider the two following statements.

(i) There exist $P, Q \in \mathbb{R}^{n \times n}$ such that

\[ P = P^T > 0, \quad Q = Q^T > 0, \]

\[ \begin{pmatrix} A^T P + PA + Q & PA_d \\ A^T_d P \end{pmatrix} < 0. \]

(ii) The following property holds:

\[ \Re \lambda(A + A_d) < 0, \quad \forall (s, z) \in \mathbb{R} \times \mathbb{C}, \quad |z| = 1 \Rightarrow \mathcal{P}(s, z) \neq 0. \]

Then, (i) implies (ii). Moreover, (ii) implies (i) if and only if

\[ \exists M \text{ invertible}, \quad \|M(sI - A)^{-1} A_d M^{-1}\|_{\infty} < 1. \]

Lemma 3 (Discrete-time positive real lemma). For any matrix $M$ fulfilling (6), the following two statements are equivalent.

(i) There exists $Q = Q^T \in \mathbb{R}^{n \times n}$, positive definite, such that

\[ M + \begin{pmatrix} A^T Q A - Q & A^T Q B \\ B^T Q A & B^T Q B \end{pmatrix} < 0. \]

(ii) $\Re \{\lambda(A)\} < 1$ and, for any $z \in \mathbb{C}$ with $|z| = 1$,

\[ \left( (sI_n - A)^{-1} B \right)^* M \left( (sI_n - A)^{-1} B \right) < 0. \]

Suppose that condition (i) of Theorem 1 holds, so there exist $P, Q \in \mathbb{R}^{n \times n}$ positive definite such that

\[ \begin{pmatrix} A^T P + PA + Q & PA_d \\ A^T_d P \end{pmatrix} < 0. \]

By use of Lemma 2, one deduces that (i) is equivalent to:

\[ \Re \lambda(A) < 0, \quad \exists Q = Q^T > 0, \forall s \in j\mathbb{R}, \]

\[ A^T_d (s^* I - A^*)^{-1} Q (sI - A)^{-1} A_d - Q < 0. \]

Applying now Lemma 3 to the previous assertion, e.g. with $B = 0_{n \times 1}$, $M = \text{diag}\{0_n, -1\}$, one deduces that $\forall (s, z) \in j\mathbb{R} \times \mathbb{C}$ with $|z| > 1$,

\[ 0 \neq \det(zI - (sI - A)^{-1} A_d) \]

\[ = \frac{z^n}{\det(sI - A)} \det(sI - A - z^{-1} A_d). \]

Changing $z$ into $z^{-1}$ and removing the irrelevant infinite values of $s$, one gets that (i) implies:

\[ \Re \lambda(A) < 0, \quad \forall (s, z) \in j\mathbb{R} \times \mathbb{C}, \quad |z| \leq 1 \Rightarrow \mathcal{P}(s, z) \neq 0. \]

It is clear that (8) implies condition (2b), which itself implies [5, Lemma 2.5]:

\[ \forall (s, z) \in j\mathbb{R} \times \mathbb{C}, \quad |z| < 1 \Rightarrow \mathcal{P}(s, z) \neq 0, \]

so one may simplify the second part of (8) to get (4b).

Now, one may prove that in (8), one may indifferently replace $\Re \lambda(A) < 0$ by $\Re \lambda(A + A_d) < 0$. Indeed, if $\Re \lambda(A) < 0$ (resp. $\Re \lambda(A + A_d) < 0$), then the polynomial map $s \mapsto \mathcal{P}(s, z)$ has only zeros with negative real part for $z = 0$ (resp. $z = 1$) and no zeros on the imaginary axis for $z \in [0, 1]$. One then concludes by a classical zeros continuity argument (see e.g. [5, Proof of Corollary 2.7]), that conditions $\Re \lambda(A) < 0$ and $\Re \lambda(A + A_d) < 0$ are interchangeable in (8). This achieves the proof of the implication (i) $\Rightarrow$ (ii). Moreover, from what precedes, one deduces
that the reverse implication \((\text{ii}) \Rightarrow (\text{i})\) holds if and only if (7) holds, that is (5). This concludes the proof of Theorem 1.

Remark that applying Lemma 3 to (8) leads to

\[
\Re \lambda(A) < 0, \quad \forall \tau \in \mathbb{R}, \quad \exists \Omega_\tau = Q_\tau^T > 0, \quad A_\tau^T (s^* I - A^T)^{-1} Q_\tau (sI - A)^{-1} A_\tau - Q_\tau < 0.
\]

This is a condition of Schur stability, which is in general weaker than the simultaneous quadratic Schur stability condition (7).

References


