A Result on Smooth Stabilization

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Abstract

We present a proof of a fundamental novel stabilization result that says that smooth stabilizability is equivalent to the existence of a submanifold transverse to the control distribution on which the feedback invariant dynamics are stable.

1 Introduction

The subject of stabilization has received attention recently and a rather limited number of conditions have been derived for a system to be stabilizable. Requiring the stabilization to be accomplished by continuous or smooth controls is more restrictive on the system, yet no general sufficient condition has been obtained so far.

In this paper, it is proved that smooth stabilizability for control systems with constant control vector fields is equivalent to the existence of stable dynamics on some manifold of dimension complementary to that of the control distribution and transverse to it. In the process, we clarify the notion of feedback invariance and we define the novel notion of control transverse dynamics. These concepts are defined independently of the datum of an output function, a fact that is not always clear in the control literature.

The approach that we find appropriate is more topological than is usual and relies on an understanding of global Lyapunov functions for dynamical systems.

2 The main result

In $\mathbb{R}^n$, we are given $m + 1$ smooth vector fields

$$f, g_1, \ldots, g_m,$$

which, in the context of controlled dynamical systems are called the state dynamics vector field and the $m$ control vector fields respectively. The $\{g_i\}$ span a distribution $D$, which is assumed to be of generic dimension $m$. It may not be integrable. We focus on global feedback control, in other words sections of $D$, or, in control language, functions $u_1, \ldots, u_m : \mathbb{R}^n \to \mathbb{R}$. The notation $\Gamma(D)$ will be used for these control sections.

The notion of state dynamics is ambiguous, since, given a control $U \in \Gamma(D)$, the vector field $X + U$ can just as well play the role of state dynamics. The appropriate global geometrical formulation that is certainly coordinate-free is to form the quotient bundle

$$\nu = T\mathbb{R}^n / D$$

which we can call the normal bundle, even though we avoid fixing specific complements to $D$ pointwise. It is a bundle of generic dimension $n - m$. From the exact sequence

$$0 \to D \to T\mathbb{R}^n \to \mathbb{R}^n / D \to 0$$

we get an exact sequence

$$0 \to \Gamma(D) \to \Gamma(T\mathbb{R}^n) \to \Gamma(\mathbb{R}^n / D) \to 0$$

and we can therefore map the vector field $f$ to a section $f'$ in $\Gamma(\nu)$; we shall use $\pi$ for the canonical projection map for the bundles and $\pi_*$ for the map of sections. This we shall call the feedback invariant form of the state dynamics. In summary, when we refer to an ‘affine control system’, we shall mean the pair

$$(D, f').$$

In this paper, we take the simplest case where $D =$constant of rank $m$. We then fix a direct sum decomposition of $T_0\mathbb{R}^n \simeq \mathbb{R}^n$:

$$\mathbb{R}^n = D^m \oplus E^{n-m}$$

(2)

and decompose the state vector $x \in \mathbb{R}^n$ as

$$x = x^1 + x^2, x^1 \in D, x^2 \in E.$$

We also assume no bounds on the control action, so all sections in $\Gamma(D)$ are available.

Definition 1. The control system $(D, f')$ is smoothly stabilizable to $x_0$ if there is a section $U \in \Gamma(D)$ such that the vector field

$$f + U$$

has a unique globally asymptotically stable equilibrium point $x_0$. 

For simplicity, we assume that $x_0 = 0$ and that any Lyapunov function for globally asymptotically stable dynamics in $\mathbb{R}^n$ with respect to 0 satisfies $V(0) = 0$.

The notion of feedback invariant dynamics is usually associated with the ‘zero dynamics’ on the zero set of an output function. However, the notion of feedback invariance is independent of output functions, as the following proposition explains:

**Proposition 1 (Control Transverse Dynamics).** Let $N^{n-m}$ be a closed submanifold of $\mathbb{R}^n$ transverse to the control distribution $D$. Then, since $TN$ gives a distribution of complements to $D$ in the trivial $n$-vector bundle over $N$, a vector field $f_N$ is defined on $N$ that is feedback invariant in the sense that the vector fields $f + U$ all map to the same $f_N$, for any $U \in \Gamma(D)$.

Thus, we have that:

**Feedback Invariant Dynamics = Control Transverse Dynamics**

and the notion of control transverse dynamics plays a crucial role in what follows.

**Proposition 2.** The set of $D$-transverse submanifolds through 0 is in one-to-one correspondence with the set of smooth functions

$$\psi : E \to D,$$

each function defining the manifold

$$\text{graph } \psi = \{(x^1, x^2) \in \mathbb{R}^n ; x^1 = \psi(x^2)\}.$$

The notion obviously has local versions by taking open subsets in $E$ containing 0.

**Theorem 1.** An affine control system with a constant control distribution $D$ is smoothly stabilizable to the origin if and only if there is a control transverse manifold $N$ with globally asymptotically stable dynamics.

**Definition 2.** The affine system $(D, f')$ is convexly stabilizable if a Lyapunov function $V$ can be found such that the sets

$$V^{-1}([0, c])$$

are convex for all $c$.

A consequence of the main theorem is the following corollary, that answers a question posed to me by Petar Kokotović.

**Corollary 1.** If the control system is smoothly stabilizable, then it is also convexly stabilizable.

**Proof of Theorem 1.** Sufficiency: This is contained in, for example, [4] and is the easier direction to prove. Essentially, one takes a Lyapunov function $V^2(x^2)$ for the feedback invariant dynamics on $N$ which is then ‘fattened up’ away from $N$ in quite an arbitrary fashion—for example, by setting

$$V(x) = V^2(x^2) + \frac{1}{2}[x^1 - \psi(x^2)]^2.$$

We omit the routine check that this indeed gives a global Lyapunov function.

**Necessity:** The proof is topological and relies on the existence of global Lyapunov functions. The set of these Lyapunov functions is quite rich: the levels can be slid along trajectories of the system in order to satisfy genericity conditions, as we shall see below.

Suppose the section $U$ gives GAS dynamics $f_U$.

**Definition 3.** The singular set of the Lyapunov function $V$ with respect to the distribution $D$ is defined by

$$\Sigma(V, D) = \{x \in \mathbb{R}^n ; dV(D) = 0\}.$$

Now we have

**Lemma 1.** In the set $\Sigma(V, D)$,

$$dV(f) < 0.$$

**Proof.** This is obvious, since

$$dV(f + U) = dV(f) < 0.$$

Now we have the following key result:

**Proposition 3.** The generic dimension of $\Sigma(V, D)$ is $n - m$.

Furthermore, a Lyapunov function $V$ can be found such that $\Sigma(V, D)$ is transverse to $D$ in an open and dense subset of $\Sigma(V, D)$.

**Proof of Proposition 3.** Fix a basis $\{e_i, i = 1, \ldots, n\}$ of $\mathbb{R}^n$ such that

$$\text{span}\{g_1, \ldots, g_m\} = \text{span}\{e_1, \ldots, e_m\}.$$

Then the set $\Sigma(V, D)$ is the set where

$$\frac{\partial V}{\partial x_1}(x_1, \ldots, x_m, x_{m+1}, \ldots, x_n) = 0$$

$$\frac{\partial V}{\partial x_m}(x_1, \ldots, x_m, x_{m+1}, \ldots, x_n) = 0$$

(3)

By the Implicit Function Theorem, we can locally solve the above implicit equation for $x^1$ in terms of $x^2$ provided the $m \times m$ block of the hessian matrix $H$:

$$H^{1,1} = \begin{pmatrix} \frac{\partial^2 V}{\partial x_1^2} & \cdots & \frac{\partial^2 V}{\partial x_1 \partial x_m} \\ \vdots & \ddots & \vdots \\ \frac{\partial^2 V}{\partial x_m \partial x_1} & \cdots & \frac{\partial^2 V}{\partial x_m^2} \end{pmatrix}$$

(4)

is nonsingular. Notice that $H^{1,1}$ is just the $mn$ principal minor of the hessian $H$ of $V$. By the Morse lemma, since there is a neighbourhood of the asymptotically stable equilibrium 0 mapped to the simple gradient vector field for the function

$$\hat{V}(y) = \frac{1}{2}||y||^2,$$

we can obviously choose the Lyapunov function $V$ to satisfy the nondegeneracy condition that $H$ be positive definite at
0. Thus $H^{1,1}$ too is positive definite at 0 and by the IFT we obtain a local graph. Extending this far away from the equilibrium requires a stronger condition on $V$. This can be accomplished by perturbing $V$ suitably:

**Lemma 2.** A Lyapunov function $V$ can be chosen for the GAS dynamics with respect to 0 that is also a Morse function and which, moreover, satisfies the condition $H^{1,1}(p)$ definite everywhere in the set

$$\Sigma(D, V).$$

except possibly on a subset of measure zero.

This result, whose proof relies on perturbing $V$ along the trajectories of the stable dynamics, but whose details we omit here, completes the proof of Proposition 3.

**Lemma 3.** On every level set of $V$ as above, the set where $H^{1,1}$ is positive definite contains a set homotopy equivalent to an $(n - m - 1)$-dimensional sphere.

**Proof.** Take

$$S^{n-m-1} = \{ \|x^2\| = 1 \} \subset \mathbb{E}^{n-m}$$

to be the unit sphere in $E$. For every $v \in S^{n-m-1}$, the ray $\mathbb{R}_+ v$ intersects the level set $V_\epsilon = V^{-1}(\epsilon)$; now find

$$\max \{ \|tv\| : tv \in V_\epsilon \}.$$

(The maximum distance to the level set in the direction $v$.) Since $H^{1,1}$ is definite almost everywhere, at the maximum it must actually be positive definite. Exercising some care patching these points together, the proof of the lemma can be completed.

The proof of the theorem is now complete, by taking the product of the spheres with the ray $\mathbb{R}_+$. The origin presents no problems, since we have used the Morse lemma to obtain a local transverse section near zero.

**Proof of Corollary 1:** Suppose graph $\psi$ gives GAS control transverse dynamics with Lyapunov function $V^2$. The main point is that, for each level of $V^2$, we can construct a convex level in $\mathbb{R}^n$ and these then form the levels of a Lyapunov function $V$ in $\mathbb{R}^n$. Controllability is not lost, since away from graph $\psi$,

$$dV(D) \neq 0$$

by construction. We omit the details (refer also to the book [1].)

### 3 Practical aspects and design methodology

#### 3.1 Dynamics on graphs

In concrete terms, since we have available the decomposition

$$\mathbb{R}^n = D_m \oplus E^{n-m}$$

we can consider systems in the ‘normal form’

$$\begin{cases}
\dot{x} &= f(x, y) \\
\dot{y} &= u
\end{cases}$$

so that the function $\psi$ gives the control-transverse dynamics

$$\dot{x} = f(x, \psi(x)).$$

The procedure is as follows:

1. First try to find a function $\psi_0$ so that the behaviour of the dynamics ‘far away’ (outside a compact set) is dissipative (compare here Sontag’s notion of input-to-state stability and its variations, [8]).

2. Then try to simplify the control-transverse dynamics by modifying the function $\psi_0$ so as to cancel unwanted dynamics. This procedure is reminiscent of bifurcation theory (and indeed there are similarities, cf. the article [2].)

If the simplification is successful, GAS dynamics are obtained on graph $\psi$ (Note: this may only exist in a neighbourhood of 0; the subsequent results are then not global.). If it is not possible to simplify, then the system is not smoothly stabilizable, and we must proceed in a different way (non-smooth, time-varying etc.), though there are no general conditions guaranteeing success, similar to the ones we have just derived for smooth stabilization.

3. A global Morse-Lyapunov function is computed. This is similar to Sontag’s CLF ([7]), except we have only considered the geometry of $D$ relative to $V$ and we have imposed some genericity assumptions on $V$.

4. The global feedback control is easily obtained using one of a number of methodologies (of $L_1$-$\tilde{g}$-type, or inverse optimal, etc.)

#### 3.2 Some examples

**Example 1.** The method described above can be applied to a number of ‘artificial’ examples in two and three dimensions to display the general geometrical features.

As an example, compare the two systems

$$\begin{cases}
\dot{x} &= \pm(x^2 - y^2(1 + y)) \\
\dot{y} &= u
\end{cases}$$

(6)
and
\[
\begin{cases}
\dot{x} &= \pm(y^2 - x^2(1 + x)) \\
\dot{y} &= \pm x
\end{cases}
\tag{7}
\]
We shall refer to them as systems \(\pm 6\) and \(\pm 7\).

**Proposition 4.** The systems \(\pm 6\) and \(\pm 7\) are globally smoothly stabilizable. On the other hand, the system \(-7\) is not globally stabilizable. It is, however, **locally** stabilizable, the region of stability being contained in the half-plane \(\{ -1 \leq x \}\).

**Proof.** First notice that in \(-7\) for \(x < -1\)
\[-(y^2 - x^2(1 + x)) < 0\]
and so no graph \(y = \psi(x)\) can give dissipative dynamics globally.

The rest of the Proposition is shown directly by examining the sets
\[O_{\pm} = \{ x : \dot{x} < 0 \} \subset \mathbb{R}^2.\]

The references [4] and [1] can be consulted for details of this method (applied to a variety of examples.)

A more realistic model is the three-dimensional Moore-Greitzer model
\[
\begin{cases}
\dot{x} &= -y + \psi(x) - 3xz \\
\dot{y} &= \frac{1}{3\sigma}(1 + x - \sqrt{\gamma_0 + u}) \\
\dot{z} &= \sigma z(1 - x^2 - z)
\end{cases}
\tag{8}
\]
(where \(\psi(x)\) is the compressor characteristic curve and \(\beta, \gamma_0\) and \(\sigma\) are constants) for which a global stabilizing controller was found (see [3] for details.)

It is instructive to also look at the example first given by Kawski [6] in the light of the above results.

**Example 2.** The two-dimensional system
\[
\begin{cases}
\dot{x} &= x - y^3 \\
\dot{y} &= u
\end{cases}
\tag{9}
\]
is known not to be smoothly stabilizable. This becomes obvious, in the light of Theorem 1, since it is not possible to find a graph \(y = \psi(x)\) on which the dynamics are GAS. The reason is that
\[\{ \dot{x} = 0 \} = \{ x = y^3 \}\]
and the slope of the inverse function \(y = x^{1/3}\) is infinite at the origin. It is checked that then every graph has nonempty intersection with the set
\[O_{\pm} = \{ x : x > 0 \},\]
since \(\psi'(0)\) is finite and hence the control transverse dynamics can never have 0 as the unique GAS equilibrium. In this case, it is argued that stabilization to 0 cannot be smooth but can be chosen to be H"older continuous.

**References**


