Keywords: Parabolic systems, regional, boundary gradient, controllability.

Abstract

The aim of this paper is to point out some new results on regional boundary gradient controllability for parabolic systems. First, we give a definition and some properties of this concept, then we concentrate on the determination of the control achieving regional boundary gradient target with minimum energy and we develop an approach that leads to numerical algorithm for the computation of the optimal control.

1 Introduction

For a given distributed parameter system, the regional controllability concept refers to control problems in which the target of interest is not fully specified as a state, but refers only to a region \( \omega \), a portion of the spatial domain \( \Omega \) on which the governing partial differential equation (S) is considered. This concept finds its applications in various industrial problems. This is the case of certain furnaces of industrial ceramic where the control of these systems consists only to maintain their temperature at a certain level in a prescribed subregion of the furnace. Many interesting results were developed in Zerrik 1993 and El Jai et al 1995. These results were extended by Zerrik et al [5, 8] to the case where \( \omega \) is a part of the boundary \( \partial \Omega \). Recently the concept of internal regional gradient controllability were developed (see Zerrik et al [6, 7]). More precisely, one is interested by steering the state gradient of the system (S) to a prescribed function only given on \( \omega \subset \Omega \). The principal reason for introducing this concept is that it provides a means of dealing with some problems from real world for example in the thermic isolation problem it may be that the control is only required to vanish the temperature gradient before crossing the brick.

In this paper one is interested in the system gradient evolution only on a boundary subregion; that is to say, we explore the possibility to reach a desired gradient in a subregion \( \Gamma \subset \partial \Omega \), we make more precise in the next section, but it is clear that the required control will depend on the considered state space and the subregion \( \Gamma \), the time \( T \) may also play a role, particularly when the controls are outside the subregion \( \Gamma \) and there is a finite speed of propagation. As a typical motivating example of this note already adduced in Zerrik et al 1998 and consists to heat a parallelepiped body \( \Omega \) so as to reach a prescribed gradient target temperature distribution on one of its face \( \Gamma \) (Fig 1.).

The paper is structured as follows. In the section 2, we present some preliminary material and mathematical concept of regional boundary gradient controllability then we give a characterization of this new concept and an example of gradient which is reachable on \( \Gamma \), but not reachable on whole domain. The third section is focused to the determination of the optimal control achieving regional boundary gradient target with minimum energy. In the fourth section we develop an approach that leads to explicit formula of the optimal control and we give numerical algorithm for the computation of the optimal control.

2 Regional Boundary Gradient Controllability

2-1 Problem statement

We start this section by presenting the notations we shall use and some preliminary material.

Let \( \Omega \) be an open bounded regular subset of \( \mathbb{R}^n \) \( \text{(} n = 2, 3 \text{)} \) with a boundary \( \partial \Omega \), \( \Gamma \) a subregion of \( \partial \Omega \) and a time interval \([0, T], T > 0\). We denote \( Q = \Omega \times [0, T], \Sigma = \partial \Omega \times [0, T] \) and we consider a parabolic system defined by

\[
\begin{align*}
\frac{\partial y}{\partial t}(x, t) &= Ay(x, t) + Bu(t) \quad Q \\
\frac{\partial y}{\partial t}(\xi, t) &= 0 \quad \Sigma \\
y(x, 0) &= y^0(x) \quad \Omega
\end{align*}
\] (2-1)

where A is a second order linear differential operator with compact resolvent and generates a strongly continuous semi-group \( (S(t))_{t \geq 0} \) on the Hilbert state space
Let $y_u(.)$ be the solution of the equation (2-1) when it is excited by the control $u$ and suppose that (2-1) has a unique solution such that $y_u(T) \in H^2(\Omega)$.

We recall that actuators form an important link between a system and its environment. A zone actuator is defined by a couple $(D, f)$, where $D \subset \Omega$ is the support of the actuator and $f$ is the spatial distribution of the action on $D$. A pointwise actuator is denoted by $(b, d)$ where $b$ designates the location of the actuator and $d$ the Dirac mass concentrated on $b$.

Let recall some definitions and results related to the internal regional gradient controllability. If $\omega \subset \Omega$ is of positive Lebesgue measure, then the following will apply

- The system (2-1) is said to be exactly gradient controllable on $\omega$ if for all $z_d \in (H^1(\omega))^n$ there exists a control $u \in U$ such that $\chi_\omega \nabla y_u(T) = z_d$.
- The system (2-1) is said to be approximately gradient controllable on $\omega$ if for all $z_d \in (H^1(\omega))^n$, given $\varepsilon > 0$, there exists a control $u \in U$ such that
  $$\|\chi_\omega \nabla y_u(T) - z_d\|_{(H^1(\omega))^n} \leq \varepsilon$$
- A sequence of actuators is said to be gradient strategic on $\omega$ if the excited system is approximately gradient controllable on $\omega$. Characterization of such actuators is given in [7].

2-2 Definition and characterization

In this subsection we define the boundary gradient controllability concept, we give a characterization and a result that links between the internal and the boundary gradient controllability.

**Definition 2.1**

The system (2-1) is said to be exactly (resp. approximately) regionally boundary gradient controllable on $\Gamma$ if for all $y_d \in (L^2(\Gamma))^n$, for all $\varepsilon > 0$, there exists a control $u \in U$ such that

$$\chi_\Gamma (\gamma \nabla y_u(T)) = y_d$$ (resp. $\|\chi_\Gamma (\gamma \nabla y_u(T)) - y_d\|_{(L^2(\Gamma))^n} \leq \varepsilon$) (2-3)

In what follows, we shall say that such system is $G$-controllable on $\Gamma$ (G for the gradient), $y_d$ will be said gradient reachable on $\Gamma$ and $(L^2(\Gamma))^n$ will be said the gradient space.

It is clear that the definitions mean that we are only interested by reaching a desired gradient on the subregion $\Gamma \subset \partial \Omega$.

**Remark 2.2**

1. The above definitions do not allow for pointwise or boundary controls since, for such systems $B \notin \mathcal{L}(\mathbb{R}^m, L^2(\Omega))$. However the extension can be carried out in similar manner if one takes regular controls such that $y_u(T) \in H^2(\Omega)$ (see El Jai and Pritchard 1988).
2. One can find states which are gradient reachable on \( \Gamma \) but not gradient reachable on the whole domain. This is illustrated with the following example.

**Counter-example**

Consider a two-dimensional system defined on \( \Omega = [0, 1] \times [0, 1] \) and described by the parabolic equation

\[
\begin{align*}
\frac{\partial z}{\partial t}(x, y, t) &= \frac{\partial^2 z}{\partial x^2}(x, y, t) \\
- \frac{\partial^2 z}{\partial y^2}(x, y, t) &= \chi_D f(x, y) u(t) \\
\end{align*}
\]

The equation (2-4) may describe a diffusion system which is excited by one actuator located in the subdomain \( D \), \( f \), the spatial distribution of the control action and \( \chi_D \) denotes the characteristic function of the domain \( D \).

Let consider \( D = [0, 1] \times [\frac{1}{2}, 1] \), \( f(x, y) = \sin 2\pi x \times e^y \), \( \Gamma = \{ (x, y) \in \partial \Omega \mid x = 0 \text{ and } 0 \leq y \leq 1 \} \) and \( g_d(x, y) = (4x^3 - (6 - \alpha)x^2 + 1) \sin 2\pi y, e^y \cos 2\pi y \) be the desired gradient on \( \Omega \) with \( \alpha = \frac{64\pi^2(e - 1)}{(1 + 4\pi^2)(1 + 4\pi^2 + 16\pi^4)} \).

**Proposition 2.3**

\( g_d \) is not gradient reachable on \( \Omega \) but is gradient reachable on \( \Gamma \).

**Proof**

1) To show that \( g_d \) is not gradient reachable on \( \Omega \) one must verify that \( g_d \notin \text{Ker} \nabla^* \nabla^* \chi_D^* \).

\[
(H^* \nabla^* g_d)(t) = \sum_{i,j=1}^{\infty} e^{\lambda_{ij}(T-t)} < \nabla^* g_d, \varphi_{ij} > < \chi_D f, \varphi_{ij} >
\]

where \( < , > \) denotes the inner product in \( H^2(\Omega) \), \( \lambda_{ij} = -(i^2 + j^2)\pi^2 \) and

\[
\varphi_{ij}(x, y) = 2a_{ij} \sin(i\pi x) \sin(j\pi y)
\]

with \( a_{ij} = (1 + (i^2 + j^2)\pi^2 + (i^4 + j^4)\pi^4)^{-\frac{1}{2}} \).

A calculation shows that

\[
< \nabla^* g_d, \varphi_{ij} > = \begin{cases} 
0 & \text{if } j \neq 2 \\
2\pi^2 a_{ij}(1 + 4\pi^2(1 + 4\pi^2)) & \\
\times \int_0^1 (-6(x^2 - x) - \alpha x + \pi e^x) \sin i\pi x dx & \text{if } j = 2 \\
\end{cases}
\]

Thus \( < \nabla^* g_d, \varphi_{ij} > < \chi_D f, \varphi_{ij} > = 0 \) if \( i \neq 2 \) or \( j \neq 2 \) and \( < \nabla^* g_d, \varphi_{22} > = 0 \).

One deduce that \( (H^* \nabla^* g_d)(t) = 0 \), hence \( g_d \) is not gradient reachable on \( \Omega \).

2) To show that \( \tilde{g}_d(x) = \chi_D \gamma g_d(x, y) \) is gradient reachable on \( \Gamma \), one must verify that \( \tilde{g}_d \notin \text{Ker} \nabla^* \nabla^* \chi_D^* \).

\[
(H^* \nabla^* \chi_D^* \tilde{g}_d)(t) = \sum_{i,j=1}^{\infty} e^{\lambda_{ij}(T-t)} < \nabla^* \chi_D^* \tilde{g}_d, \varphi_{ij} > < \chi_D f, \varphi_{ij} >
\]

The calculation gives

\[
(H^* \nabla^* \chi_D^* \tilde{g}_d)(t) = e^{\lambda_{22}(T-t)} < \nabla^* \chi_D^* \tilde{g}_d, \varphi_{22} >
\]

It is easy to verify that \( < \nabla^* \chi_D^* \tilde{g}_d, \varphi_{22} > \neq 0 \) and \( < \chi_D f, \varphi_{22} > \neq 0 \).

Therefore \( \tilde{g}_d \) is gradient reachable on \( \Gamma \).

**Proposition 2.4**

1) The system (2-1) is exactly G-controllable on \( \Gamma \) if and only if there exists \( c > 0 \), such that \( \forall y^* \in (L^2(\Gamma))^n \)

\[
\|y^*\|_{(L^2(\Gamma))^n} \leq c \left\| B^* S^*(T - t) \nabla^* \gamma \chi_D^* y^* \right\|_U
\]

2) If the semi-group \( (S(t))_{t \geq 0} \) is analytic, then the system (2-1) is approximately G-controllable on \( \Gamma \) if and only if

\[
\bigcup_{n \in \mathbb{N}} \left\{ \left( \chi_D^* \nabla A^n S(t) B U \right) = (L^2(\Gamma))^n \right\} \forall t \in [0, T].
\]
Proof

1) Let $h = Id_{(L^2(\Gamma))^n}$ and $g = \chi_\gamma \nabla H$, the system (2-1) is exactly G-controllable if $Im h \subset Im g$ which is equivalent to, there exists $c > 0$ such that $\|h^* z\|_{L^2(\Gamma)^n} \leq c \|g^* z\|_U, \forall z \in (L^2(\Gamma))^n$, which gives the considered inequality.

2) If the system (2-1) is approximately G-controllable on $\Gamma$ we have $Im (\chi_\gamma \nabla S(\xi)) (L^2(\Gamma))^n$ which is equivalent to $\{z, \chi_\gamma \nabla S(\xi) B v \neq 0 \forall s \in [0, T], \forall v \in U\} \Rightarrow z = 0$.

Assume that there exists $s \in [0, T]$ such that

\[ \bigcup_{n \geq 0} Im (\chi_\gamma \nabla A^n S(\xi) B) \neq (L^2(\Gamma))^n \]

then there exists $z_0 \neq 0, z_0 \in (L^2(\Gamma))^n$ and $s \in [0, T]$ such that

\[ < z_0, \chi_\gamma \nabla A^n S(\xi) B v > = 0 \quad \forall \ n \in \mathbb{N} \quad \forall v \in U \]

or equivalently there exists $z_0 \neq 0$ and $s \in [0, T]$ such that

\[ \frac{d^n}{ds^n} < z_0, \chi_\gamma \nabla S(\xi) B v > = 0 \quad \forall \ n \in \mathbb{N} \quad \forall v \in U \]

by analyticity we have

\[ < z_0, \chi_\gamma \nabla S(t) B v > = 0 \quad \forall v \in U \text{ and } t \text{ close to } s \text{ i.e. the system is not approximately G-controllable on } \Gamma. \]

Conversely, suppose that the system (2-1) is not approximately G-controllable on $\Gamma$, there exists $z_0 \neq 0$ such that

\[ < z_0, \chi_\gamma \nabla S(t) B v > = 0 \quad \forall t \in [0, T], \forall v \in U \text{ and } \forall n \in \mathbb{N} \]

then $\frac{d^n}{ds^n} < z_0, \chi_\gamma \nabla S(t) B v > = 0 \quad \forall t \in [0, T]$ and consequently

\[ < z_0, \chi_\gamma \nabla A^n S(t) B U > = 0 \quad \forall n \in \mathbb{N}, \forall t \in [0, T], \forall v \in U. \]

That is to say

\[ \bigcup_{n \geq 0} (\chi_\gamma \nabla A^n S(t) B U) \neq (L^2(\Gamma))^n, \]

which is a contradiction.

Now, let $\omega \subset \Omega$ such that $\Gamma = \partial \omega \cap \partial \Omega$. The following result gives a link between regional internal gradient controllability and regional boundary gradient controllability. Here we assume that the gradient space is $(H^{\frac{1}{2}}(\Gamma))^n$.

Proposition 2.5

If the system (2-1) is exactly (resp.approximately) gradient controllable on $\omega$, then it is exactly (resp.approximately) G-controllable on $\Gamma$.

Proof

For all $g = (g_1, \ldots, g_n) \in (H^{\frac{1}{2}}(\partial \Omega))^n$ we consider

$\tilde{R} g = (R g_1, \ldots, R g_n)$ where $R : H^{\frac{1}{2}}(\partial \Omega) \rightarrow H^1(\Omega)$ is a linear continuous operator such that $\gamma_0 R y = y, \forall y \in H^{\frac{1}{2}}(\partial \Omega)$ (see Dautray-Lions 1988).

Let $g_d = (g_{d_1}, \ldots, g_{d_n}) \in (H^{\frac{1}{2}}(\Gamma))^n$, we denote by $\tilde{g}_d = (\tilde{g}_{d_1}, \ldots, \tilde{g}_{d_n})$ the extension of $g_d$ to $\partial \Omega$ such that $\tilde{g}_d \in (H^{\frac{1}{2}}(\partial \Omega))^n$, then by trace theorem, there exists $\tilde{R} \tilde{g}_d \in (H^1(\Omega))^n$ with a bounded support such that $\gamma_1 \tilde{R} \tilde{g}_d = \tilde{g}_d$.

Let $\tilde{\chi}_\gamma$ be the map restriction from $(H^{\frac{1}{2}}(\partial \omega))^n \rightarrow (H^{\frac{1}{2}}(\Gamma))^n$ and $\tilde{\gamma}$ be the trace mapping from $(H^1(\omega))^n \rightarrow (H^{\frac{1}{2}}(\partial \omega))^n$.

As the system (2-1) is exactly gradient controllable on $\omega$, then there exists a control $u \in U$ such that $\chi_\omega \nabla y_u(T) = \chi_\omega R \tilde{g}_d$ thus $\tilde{\chi}_\gamma (\tilde{\gamma} \chi_\omega \nabla y_u(T)) = g_d$. Consequently the system (2-1) is exactly G-controllable on $\Gamma$.

Now, if the system (2-1) is approximately gradient controllable on $\omega$, for all $\varepsilon > 0$, there exists $u \in U$ such that $\|\chi_\omega \nabla y_u(T) - \chi_\omega R \tilde{g}_d\|_{(H^1(\omega))^n} \leq \varepsilon$ and by continuity of the trace mapping $\tilde{\gamma}$, we have $\|\tilde{\gamma} (\chi_\omega \nabla y_u(T)) - \tilde{\gamma} (\chi_\omega R \tilde{g}_d)\|_{(H^{\frac{1}{2}}(\partial \omega))^n} \leq \varepsilon$ therefore $\|\tilde{\chi}_\gamma (\chi_\omega \nabla y_u(T)) - g_d\|_{(H^{\frac{1}{2}}(\Gamma))^n} \leq \varepsilon$.

Consequently the system (2-1) is approximately G-controllable on $\Gamma$.

3 Boundary gradient target control with minimum energy

In this section, we explore the possibility of finding a minimum energy control which ensures the transfer of the system (2-1) to a desired gradient $g_d$ on the boundary subregion $\Gamma$. The minimum energy regional boundary gradient control problem may be stated by considering the natural minimization problem

\[
\left\{ \begin{array}{l}
\min_{u \in U_{ad}} J(u) = \int_0^T \|u(t)\|^2_{g_m} dt \\
\end{array} \right.
\]

(3-1)

where $U_{ad} = \{u \in U \mid \chi_\gamma \nabla y_u(T) = g_d\}$. This problem is solved by two approaches.

Let $R_\gamma = (\chi_\gamma \nabla H) (\chi_\gamma \nabla H)^*$ and $y_d = (g_d - \chi_\gamma \nabla S(T) y_n)^*$.

3-1 Direct approach

If $g_d \in (L^2(\Gamma))^n$, then the solution of the problem (3-1) is given by the following result
Proposition 3.1
If the system (2-1) is approximately G-controllable on \( \Gamma \) and \( y_d \in \text{Im} R \), then the problem (3-1) has only one solution given by
\[
    u^*(t) = (\chi \gamma \nabla H)^* R^{-1} y_d \tag{3-2}
\]

**Proof**

The solution of the equation (2-1) excited by the control \( u \) is given by \( y_d(t) = S(t) y^0 + H u \).
For all \( u \in U_{ad} \), we have \( \chi \gamma \nabla H u = y_d \).
To the problem (3-1) we associate the Lagrange multiplier \( L(u, \lambda) = \| u \|^2 + < \chi \gamma \nabla H u - y_d, \lambda >_{(L^2(\Gamma))^n}, \lambda \in (L^2(\Gamma))^n \).
If \( u^* \) minimizes \( L \) then \( \frac{\partial L}{\partial u}(u^*, \lambda) = 0 \), which gives
\[
    2 u^* + (\chi \gamma \nabla H)^* \lambda = 0,
\]
or equivalently \( u^* = -\frac{1}{2}(\chi \gamma \nabla H)^* \lambda \). Since \( u^* \in U_{ad} \) we obtain
\[
    -\frac{1}{2} R \lambda = y_d.
\]

As the system (2-1) is approximately G-controllable on \( \Gamma \), we have \( U(\chi \gamma \nabla H) = (L^2(\Gamma))^n \) then \( R \) is one to one and as \( y_d \in \text{Im} R \), we have \( \lambda = -2 R^{-1} y_d \).
Therefore \( u^* = (\chi \gamma \nabla H)^* R^{-1} y_d \), \( U_{ad} \) is closed convex and \( J \) is a strict convex function then the optimal control \( u^* \) solution of the problem (3-1) is unique.

3-2 Internal regional gradient controllability approach

Let us consider the system (2-1) and assume that its solution \( y_u(\cdot) \) is such that \( y_u(T) \in H^3(\Omega) \) and consider the problem (3-1). Here we assume that
\[
    g_d = (g_{d1}, \ldots, g_{dn}) \in (H^2(\Gamma))^n.
\]
Let \( p \geq 0 \) integer, \( F_p = \bigcup_{z \in P} B(z, \frac{1}{p}) \) and \( \omega_p = F_p \cap \Omega \)
where \( B(z, \frac{1}{p}) \) is the open ball of radius \( \frac{1}{p} \) and center \( z \).

Consider the problem
\[
    \begin{cases}
    \min J(u) = \int_0^T \| u(t) \|^2_{L^2(\Omega)} \, dt \\
    u \in U_{ad}^p
    \end{cases}
\]
with \( U_{ad}^p = \{ u \in U \mid \chi \omega_p \Delta y_u(T) = \chi \omega_p z_d \} \)
and \( z_d = (z_{d1}, \ldots, z_{dn}) \in (H^2(\Omega))^n \) with \( z_{di} \) is the solution of the equation
\[
    \begin{cases}
    \Delta z = 0 \quad \Omega \\
    z = \tilde{g}_{di} \quad \partial \Omega
    \end{cases}
\]
where \( \Delta \) is the Laplace operator and \( \tilde{g}_{di} \) is the extension of \( g_{di} \) in \( H^\frac{3}{2}(\partial \Omega) \). Then the problem of reaching \( g_d \) on \( \Gamma \)
may be solved by reaching \( z_d \) on \( \omega_p \).

We denote \( R_{\omega_p} = (\chi \omega_p \nabla H) (\chi \omega_p \nabla H)^* \),
\[
    \tilde{z}_p = \chi \omega_p (z_d - \nabla S(T) y^0).
\]

Proposition 3.2
If the system (2-1) is approximately gradient controllable on \( \omega_p \) and \( z_p \in \text{Im} R_{\omega_p} \), then the problem (3-3) has a unique solution given by
\[
    u^*_p(t) = (\chi \omega_p \nabla H)^* R_{\omega_p}^{-1} \tilde{z}_p \tag{3-5}
\]
and \( u^*_p \) converges weakly to \( u^* \) solution of the problem (3-1).

**Proof**

1) The problem (3-3) has only one solution given by (3-5) (the proof is similar to that given for the proposition (3-1)).
2) Consider the problem
\[
    \begin{cases}
    \min J(v) = \int_0^T \| v(t) \|^2_{L^2(\Omega)} \, dt \\
    v \in V_{ad} = \{ v \in U \mid \nabla y_v(T) = z_d \}
    \end{cases}
\]
we have \( V_{ad} \subset U_{ad}^p \) and \( \min_{u \in U_{ad}^p} \| u \|^2 \leq \min_{v \in V_{ad}} \| v \|^2 \)
then \( \| u^*_p \| \leq \| u^* \| \), where \( u^*_p \) is the unique solution of (3-3) and \( u^* \) is the unique solution of (3-6). One deduce that the suite \( (u^*_p)_p \) is bounded in \( U \).
As \( U \) is a reflexive space, there exists a subsequence \( (u_{p_k})_k \)
which converges weakly to \( u^* \) in \( U \).
Since the operator \( \nabla H \) is linear and continuous from \( U \rightarrow (H^2(\Omega))^n \), then the sequence \( (\nabla H u_{p_k})_k \) converges weakly to \( \nabla H u^* \).

The injection of \( (H^2(\Omega))^n \) in \( C^0(\bar{\Omega}) \) (the space of function restriction to \( \Omega \) of continuous function of \( B^m \)) is compact, hence \( (\nabla H u_{p_k})_k \) converges strongly to \( \nabla H u^* \) in \( C^0(\bar{\Omega}) \)
and by (3-3) we have
\[
    \chi \omega_p \nabla H u_{p_k} = \chi \omega_p (z_d - \nabla S(T) y^0) \quad C^0(\bar{\Omega}).
\]
The sequence \( u_{p_k} \) converges to \( \Gamma = \bigcap_{1}^{p} \omega_p \) in Kuratowski’s sens, so \( (\nabla H u_{p_k})_k \subset \{ z_d - \nabla S(T) y^0 \}_k \),
therefore \( (\nabla H u_{p_k})_k \subset \{ z_d - \nabla S(T) y^0 \}_k \), then \( u^* \) is solution of the equation \( \chi \gamma \nabla H u = y_d \).
From proposition (3.1) this equation has only one solution given by
\[
    u^* = (\chi \gamma \nabla H)^* R_{\Gamma}^{-1} y_d
\]
and which is of minimum energy.

4 Numerical approach

We have seen that \( u^*_p \) approximate \( u^* \) the optimal control that steers the system to a given regional boundary gradient. In this section, we explore an approach devoted to the calculation of \( u^*_p \) solution of the problem (3-3), the method is based on an extension of the internal regional gradient controllability. Consider the system (2-1) with \( A \) assumed to be of constant coefficients.

Let
\[
    \tilde{G} = \{ g \in L^2(\Omega) \text{ such that } g = 0 \text{ on } \Omega \setminus \omega_p \}.
\]
4-1 Case of zone actuator

If we consider the system (2-1) excited by a control applied by mean of a zone actuator \((D, f)\) where \(D \subset \Omega\) is the actuator support and \(f \in L^2(D)\) defines the spatial distribution of the control on \(D\), then we have \(Bu(t) = \chi_D f(x) u(t)\) and the system may be written in the form

\[
\begin{aligned}
\begin{cases}
\frac{\partial y}{\partial t}(x, t) &= Ay(x, t) + \chi_D f(x) u(t) & Q \\
y(\xi, t) &= 0 & \Sigma \\
y(x, 0) &= y^o(x) & \Omega
\end{cases}
\end{aligned}
\tag{4-1}
\]

For \(\varphi_o \in \hat{G}\), consider the system

\[
\begin{aligned}
\begin{cases}
\frac{\partial \varphi_o}{\partial t}(x, t) &= -A^* \varphi_o(x, t) & Q \\
\varphi_o(\xi, t) &= 0 & \Sigma \\
\varphi_o(x, T) &= \varphi_o(x) & \Omega
\end{cases}
\end{aligned}
\tag{4-2}
\]

and assume it has a unique solution \(\varphi_o \in L^2(0, T, H^2_0(\Omega)) \cap C^1(\Omega \times [0, T])\).

For a given \(\varphi_o \in \hat{G}\), we consider the system (4-2) and define the mapping

\[
\begin{aligned}
\varphi_o \in \hat{G} \mapsto \|\varphi_o\|_{\hat{G}}^2 &= \int_0^T \left( \sum_{i=1}^n < f, \frac{\partial \varphi_o}{\partial x_i} >_{L^2(D)} \right)^2 dt 
\end{aligned}
\tag{4-3}
\]

which is a norm on \(\hat{G}\) if the actuator \((D, f)\) is gradient strategic on \(\omega_p\) (see Zerrik et al 1999 [6]).

We denote the completion of the set \(\hat{G}\) with respect to the norm (4-3) again by \(\hat{G}\) and consider the system

\[
\begin{aligned}
\begin{cases}
\frac{\partial \Psi}{\partial t}(x, t) &= A\Psi(x, t) \\
&+ \sum_{i=1}^n < f, \frac{\partial \varphi_o}{\partial x_i} >_{L^2(D)} \chi_D & Q \\
\Psi(\xi, t) &= 0 & \Sigma \\
\Psi(x, 0) &= y^o(x) & \Omega
\end{cases}
\end{aligned}
\tag{4-4}
\]

which may be decomposed in two the following ones

\[
\begin{aligned}
\begin{cases}
\frac{\partial \Psi_1}{\partial t}(x, t) &= A\Psi_1(x, t) & Q \\
\Psi_1(\xi, t) &= 0 & \Sigma \\
\Psi_1(x, 0) &= y^o(x) & \Omega
\end{cases}
\end{aligned}
\tag{4-5}
\]

and

\[
\begin{aligned}
\begin{cases}
\frac{\partial \Psi_2}{\partial t}(x, t) &= A\Psi_2(x, t) \\
&+ \sum_{i=1}^n < f, \frac{\partial \varphi_o}{\partial x_i} >_{L^2(D)} f(x) \chi_D & Q \\
\Psi_2(\xi, t) &= 0 & \Sigma \\
\Psi_2(x, 0) &= 0 & \Omega
\end{cases}
\end{aligned}
\tag{4-6}
\]

Let \(\wedge\) be the operator defined by

\[
\begin{aligned}
\wedge : \hat{G} &\longrightarrow \hat{G}^* \\
\varphi_o &\longrightarrow \tilde{P} \left( \sum_{i=1}^n \frac{\partial \Psi_2}{\partial x_i}(T) \right)
\end{aligned}
\tag{4-7}
\]

while \(\tilde{P} \left( \sum_{i=1}^n \frac{\partial \Psi_2}{\partial x_i}(T) \right) = \begin{cases} -\sum_{i=1}^n \frac{\partial \Psi_2}{\partial x_i}(T) & \omega_p \\
0 & \Omega \setminus \omega_p \end{cases}\)

and \(\hat{G}^*\) is the dual of \(\hat{G}\).

With these notations the regional gradient control problem on \(\omega_p\) leads to the resolution of the equation

\[
\begin{aligned}
\wedge \varphi_o = \tilde{P} \left( \sum_{i=1}^n ((z_d) - \frac{\partial \Psi_1}{\partial x_i}(T)) \right)
\end{aligned}
\tag{4-8}
\]

We have the following result

**Proposition 4.1**

If the actuator \((D, f)\) is gradient strategic on \(\omega_p\), then the equation (4-8) has a unique solution \(\varphi_o \in \hat{G}\) and the control \(u_p^*(t) = < F, \nabla \varphi_p >_{(L^2(D))^n}\) is the solution of the problem (3-3), where \(F = (f, f, ..., f)\).

**Proof**

First we prove that the equation (4-8) has a unique solution, for that purpose multiply the equation (4-6) by \(\frac{\partial \varphi_o}{\partial x_i}\) and integrate on \(Q\), we have

\[
\begin{aligned}
\int_Q \frac{\partial \varphi_o}{\partial x_i}(t) \Psi_2(t) dt dx &= \int_Q \frac{\partial \varphi_o}{\partial x_i}(t) A\Psi_2(t) dt dx \\
&+ \int_Q \frac{\partial \varphi_o}{\partial x_i}(t) \chi_D f < F, \nabla \varphi_p >_{(L^2(D))^n} dt dx
\end{aligned}
\]

which gives

\[
\begin{aligned}
\int_\Omega \left[ \frac{\partial \varphi_o}{\partial x_i}(t) \right]_0^T dx - \int_Q \left( \frac{\partial \varphi_o}{\partial x_i} \right)'(t) \Psi_2(t) dt dx &= \int_Q \frac{\partial \varphi_o}{\partial x_i}(t) A\Psi_2(t) dt dx \\
&+ \int_0^T < f, \frac{\partial \varphi_o}{\partial x_i} >_{L^2(D)} \times < F, \nabla \varphi_p >_{(L^2(D))^n} dt
\end{aligned}
\]


thus
\[
\int_Ω \left( \frac{∂φ_p(T)}{∂x_i}(T)Ψ_2(T) - \frac{∂φ_p(0)}{∂x_i}(0)Ψ_2(0) \right) dx = \\
\int_Q \left[ \frac{∂φ_p}{∂x_i}(t)AΨ_2(t) - Ψ_2(t)A^*\frac{∂φ_p}{∂x_i}(t) \right] dt dx \\
+ \int_0^T < f, \frac{∂φ_p}{∂x_i} >_{L^2(D)} < F, \nabla φ_p >_{(L^2(D))^n} dt
\]
By Green formula, we obtain
\[
- \int_Ω \varphi_0 \frac{∂Ψ_2}{∂x_i}(T) dx = \\
\sum_t \left( \frac{∂φ_p}{∂x_i}(t) - Ψ_2(t) \frac{∂}{∂v_A^*}(\frac{∂φ_p}{∂x_i}(t)) \right) dt dx \\
+ \int_0^T < f, \frac{∂φ_p}{∂x_i} >_{L^2(D)} < F, \nabla φ_p >_{(L^2(D))^n} dt
\]
and with the boundary conditions, we have
\[
- \int_Ω \varphi_0 \frac{∂Ψ_2}{∂x_i}(T) dx = \\
\int_0^T < f, \frac{∂φ_p}{∂x_i} >_{L^2(D)} < F, \nabla φ_p >_{(L^2(D))^n} dt
\]
Then
\[
< \varphi_0, φ_0 > = \int_0^T \left( \sum_{i=1}^n < f, \frac{∂φ_p}{∂x_i} >_{L^2(D)} \right)^2 dt \\
= ||φ_0||^2_G
\]
Hence ∧ is one to one and the equation (4-8) has a unique solution.

Now, for v ∈ U_a^p, we have
\[
< \frac{∂φ_p}{∂x_i}(T), y_0(T) - y_{u_p}(T) >_{L^2(Ω)} \\
- < \frac{∂φ_p}{∂x_i}(0), y_0(0) - y_{u_p}(0) >_{L^2(Ω)} \\
= \int_Ω \left( \frac{∂φ_p}{∂x_i}(t)(y_0(t) - y_{u_p}(t)) \right) dt dx \\
+ \int_0^T \left( \frac{∂φ_p}{∂x_i}(t)(y_0(t) - y_{u_p}(t)) \right) dt dx \\
- \int_Ω \left( \frac{∂φ_p}{∂x_i}(t)(y_0(t) - y_{u_p}(t)) \right) dt dx \\
+ \int_0^T \left( \frac{∂φ_p}{∂x_i}(t)(v - u_p^*) \right) dt dx
\]
By Green formula we obtain
\[
< \frac{∂φ_p}{∂x_i}(T), y_0(T) - y_{u_p}(T) >_{L^2(Ω)} \\
- < \frac{∂φ_p}{∂x_i}(0), y_0(0) - y_{u_p}(0) >_{L^2(Ω)} \\
= \int_Ω \left( \frac{∂φ_p}{∂x_i}(t)(y_0(t) - y_{u_p}(t)) \right) dt dx \\
- \int_Ω \left( \frac{∂φ_p}{∂x_i}(t)(y_0(t) - y_{u_p}(t)) \right) dt dx \\
+ \int_0^T < f, \frac{∂φ_p}{∂x_i} >_{L^2(D)} < F, \nabla φ_p >_{(L^2(D))^n} dt
\]
The initial and boundary conditions give
\[
< φ_0, \frac{∂}{∂x_i}(y_0(T) - y_{u_p}(T)) >_{L^2(Ω)} \\
= - \int_0^T < f, \frac{∂φ_p}{∂x_i} >_{L^2(D)} < F, \nabla φ_p >_{(L^2(D))^n} dt
\]
By sommation we obtain
\[
- \sum_{i=1}^n < φ_0, \frac{∂}{∂x_i}(y_0(T) - y_{u_p}(T)) >_{L^2(Ω)} \\
= \int_0^T < f, \frac{∂φ_p}{∂x_i} >_{L^2(D)} < F, \nabla φ_p >_{(L^2(D))^n} dt
\]
Hence J'(u_p^*)(v - u_p^*) = 0 and this establishes the optimality of u_p^*. ■

4-2 Case of pointwise actuator

Let us consider the system (2-1) with a one pointwise internal control applied on b ∈ Ω, where b designates the location of the actuator. The system (2-1) may be rewritten in the form
\[
\begin{cases}
\frac{∂y}{∂t}(x, t) = Ay(x, t) + δ(x - b)u(t) & \text{Q} \\
y(x, t) = 0 & \text{Σ} \\
y(x, 0) = y^0(x) & \Omega
\end{cases}
\]
We assume that the solution of (4-9) is such that
\[y_0(T) ∈ H^3(Ω)\]
Now, for a given φ_0 ∈ G̃, we consider the system (4-2) and define the mapping
\[φ_0 ∈ G̃ → ||φ_0||^2_G = \int_0^T \left( \sum_{i=1}^n \frac{∂φ_p}{∂x_i}(b, t) \right)^2 dt \]
which is a semi-norm on G̃ and consider the system
\[
\begin{cases}
\frac{∂y}{∂t}(x, t) = AΨ(x, t) + \sum_{i=1}^n \frac{∂φ_p}{∂x_i}(b, t)δ(x - b) & \text{Q} \\
Ψ(x, t) = 0 & \text{Σ} \\
Ψ(x, 0) = y^0(x) & \Omega
\end{cases}
\]
For φ_0 ∈ G̃ the system (4-2) gives φ_p and the system (4-11) gives Ψ(T).
We can consider a decomposition of the above system (4-11) in the two systems.
(4-14) has a unique solution

\[ \Psi (\xi, t) = 0 \quad \Sigma \]

and

\[
\begin{align*}
\frac{\partial \Psi_1}{\partial t} (x, t) &= A \Psi_1 (x, t) \quad Q \\
\Psi (\xi, t) &= 0 \quad \Sigma \\
\Psi_1 (x, 0) &= y^\circ (x) \quad \Omega 
\end{align*}
\] (4-12)

For \( \varphi_i \in \mathcal{G} \), let \( \wedge \) be the operator defined by equation (4-7) with \( \Psi_2 \) is the solution of equation (4-13). Then the regional gradient controllability problem on \( \omega_p \) is equivalent to the resolution of the equation

\[ \wedge \varphi_0 = \tilde{P} \left( \sum_{i=1}^{n} (z_d)_i \right) - \frac{\partial \Psi_1}{\partial x_i} (T) \] (4-14)

and we have the result

**Proposition 4.2**

If the actuator \((b, \delta_b)\) is gradient strategic on \( \omega_p \), the equation (4-14) has a unique solution \( \varphi_0 \in \mathcal{G} \) and the control steering the system to the desired gradient \( z_d \) on \( \omega_p \) with minimum energy is given by \( u^*_p (t) = \sum_{i=1}^{n} \frac{\partial \varphi_p}{\partial x_i} (b, t) \).

With some minor technical differences, the proof is similar to the zone case.

### 4-3 Numerical algorithm

We have seen that the solution of the regional gradient controllability problem is obtained by the solution of the equation

\[ \wedge \varphi_0 = \tilde{P} \left( \sum_{k=1}^{n} (z_d)_k \right) - \frac{\partial \Psi_1}{\partial x_k} (T)) \] (4-15)

The numerical approach may be achieved very easily if one can calculate in a suitable basis \( (\varphi_i) \), the components \( \wedge_{ij} \) of \( \wedge \). The problem will be approximated by the solution of the linear system

\[ \sum_{j=1}^{M} \wedge_{ij} \varphi_{0,j} = h_i, \quad i = 1, M \] (4-16)

where \( M \) is the order of approximation and \( h_i \) are the components of \( h = \tilde{P} \left( \sum_{k=1}^{n} (z_d)_k \right) - \frac{\partial \Psi_1}{\partial x_k} (T)) \).

\( (\varphi_i) \) a basis of eigenfunctions of \( A^* \) in \( H^1 (\Omega) \), associated to the eigenvalues \( \lambda_i \) assumed to be of multiplicity one. In the pointwise case we have

\[ \frac{\partial \varphi^2_p (b, t)}{\partial x_k} = \sum_{j=1}^{\infty} e^{\lambda_j (T-t)} < \varphi_0, \varphi_j >_{L^2 (\omega_p)} \frac{\partial \varphi^2_p (b)}{\partial x_k} \]

and then

\[ < \wedge \varphi_0, \varphi_0 > = \sum_{i,j=1}^{\infty} \frac{\partial \varphi_i}{\Delta x_k} (b) \frac{\partial \varphi_j}{\Delta x_l} (b) \]

Thus, the components of \( \wedge \) are given by

\[ \wedge_{ij} = \frac{(e^{(\lambda_i + \lambda_j)T} - 1)}{(\lambda_i + \lambda_j)} \sum_{k,l=1}^{n} \frac{\partial \varphi_i}{\Delta x_k} (b) \frac{\partial \varphi_j}{\Delta x_l} (b) \]

**Remark 4.3**

1. In the zone case the same developments as the pointwise case lead to the components of \( \wedge \)

\[ \wedge_{ij} = \frac{(e^{(\lambda_i + \lambda_j)T} - 1)}{(\lambda_i + \lambda_j)} \sum_{k,l=1}^{n} \frac{\partial \varphi_i}{\Delta x_k} (b) \frac{\partial \varphi_j}{\Delta x_l} (b) \]

2. In the case of many pointwise actuators \((b_q, q=1, r)\) the components of \( \wedge \) are given by

\[ \wedge_{ij} = \frac{(e^{(\lambda_i + \lambda_j)T} - 1)}{(\lambda_i + \lambda_j)} \sum_{q=1}^{r} \sum_{k,l=1}^{n} \frac{\partial \varphi_i}{\Delta x_k} (b_q) \frac{\partial \varphi_j}{\Delta x_l} (b_q) \]

3. In general it is not easy to calculate the eigenfunctions of the operator \( A^* \), a direct approach allows to overcome this difficulty and leads to the desired gradient on \( \omega_p \). So the problem turns up naturally to find \( \varphi_0 \) which is the solution of the following problem

\[ \begin{cases} 
\text{Min} \| X_{\omega_p} \nabla y^* (T) - z_d \|_{(H^2 (\omega_p))}^n \\
\varphi_0 \in \mathcal{G} 
\end{cases} \] (4-17)

The resolution of this problem can easily be achieved by the direct minimization algorithm

1. **Initial data** \( p, q_d, \text{ actuator}, \epsilon \)
2. **Solve the equation (3-4) \( \rightarrow z_d \)**
3. **Choose** \( \varphi_0 \text{ in } \mathcal{G} \)
4. **Solve the system (4-2) \( \rightarrow \sum_{i=1}^{n} \frac{\partial \varphi_p}{\partial x_i} (b, t) \)**
5. **Solve the system (4-12) \( \rightarrow \Psi_1 \)**
6. **Solve the system (4-13) \( \rightarrow \Psi_2 \)**
7. **If** \( \| X_{\omega_p} \nabla \Psi (T) - z_d \|_{(H^2 (\omega_p))}^n > \epsilon \) **go to step 3**
8. **The optimal control is given by** \( u^*_p = \sum_{i=1}^{n} \frac{\partial \varphi_p}{\partial x_i} (b, t) \)
4. The developed method steers the system to the desired gradient on the subregion $\bar{\omega}_p$ and the residual gradient on $\Omega \setminus \bar{\omega}_p$ will depend on the applied control.

5 Conclusion

In this paper we have developed an extension of the results on regional gradient controllability to a realistic situation encountered in various applications where the control must achieve a certain objective on the boundary of the geometric domain where the system is considered. Moreover we have explored an approach which allows the implementation of such control. In addition if there is a minimum energy control law, how does this depend on $\Gamma$ and the location of the controls and in particular, is it possible to optimize the locations?

References


