Homogeneous systems: stability, boundedness and duality

Luc Moreau†, Dirk Aeyels†, Joan Peuteman†, Rodolphe Sepulchre‡

†SYSTeMS
Ghent University
Technologiepark 9
9052 Zwijnaarde, Belgium
‡Institut Montefiore, B28
University of Liege
4000 Liege Sart-Tilman, Belgium
luc.moreau@rug.ac.be

Keywords: stability, time-varying, homogeneity, geometry, boundedness, averaging.

Abstract

We introduce a duality principle for homogeneous vectorfields. As an application of this duality principle, stability and boundedness results for negative order homogeneous differential equations are obtained, starting from known results for positive order homogeneous differential equations.

1 Introduction

Homogeneous vectorfields are vectorfields possessing a symmetry with respect to a family of dilations. They play a prominent role in various aspects of nonlinear control theory. See, for example, [1, 2, 5, 8] for some applications in feedback control.

Recently, interesting results have been obtained for the particular class of positive order homogeneous differential equations: Peuteman and Aeyels [9] have proven that a time-varying positive order homogeneous differential equation is asymptotically stable if the associated averaged differential equation is asymptotically stable; Peuteman, Aeyels and Sepulchre [10] have proven that a time-varying positive order homogeneous differential equation is bounded if each associated time-frozen differential equation is bounded. These results allow to reduce a stability and boundedness analysis of a time-varying differential equation to an analysis of time-invariant differential equations, possibly resulting in an important simplification. The proofs of these results exploit the inherently slow character of solutions of positive order homogeneous differential equations near the origin, and the inherently fast character far away from the origin.

In the present paper, we prove the dual results for negative order homogeneous differential equations: a time-varying negative order homogeneous differential equation is asymptotically stable if each time-frozen differential equation is asymptotically stable; it is bounded if the averaged is bounded. We provide an indirect proof for these results, obtaining them as corollaries of the above-mentioned results for positive order homogeneous differential equations: we first introduce a duality principle for homogeneous vectorfields; this duality then provides an elegant way to pass from the results for positive order homogeneous differential equations to the dual results for negative order homogeneous differential equations.

This paper is organized as follows. Having introduced the preliminaries in Section 2, we define in Section 3 a duality transformation between positive and negative order homogeneous vectorfields. We start Section 4 with recalling a stability and boundedness result for positive order homogeneous differential equations. Then, as an application of the duality concept, we obtain the dual results for negative order homogeneous differential equations. Section 5 concludes the paper.

For proofs, additional details and further results we refer to the journal version of this paper [7].

2 Preliminaries

2.1 Time-varying vectorfields and differential equations

This paper is concerned with differential equations on $\mathbb{R}_0^n := \mathbb{R}^n \setminus \{0\}$. Notice that we exclude the origin from the state space. Although not standard, this seems to be natural within the study of homogeneous vectorfields; it allows to avoid complications that would otherwise arise from singularities at the origin. We emphasize that results obtained in this framework may be reinterpreted in terms of differential equations on the complete state-space $\mathbb{R}^n$.

We take $\mathbb{R}_0^n$ with the relative topology induced by the usual topology on $\mathbb{R}^n$. Let $\mathcal{F}(\mathbb{R}_0^n)$ (resp. $\mathcal{X}(\mathbb{R}_0^n)$) be the set of all real-valued functions $f : \mathbb{R} \times \mathbb{R}_0^n \rightarrow \mathbb{R} : (t, x) \mapsto f(t, x)$ (resp. the set of all vector-valued maps $X : \mathbb{R} \times \mathbb{R}_0^n \rightarrow \mathbb{R}^n : (t, x) \mapsto X(t, x)$) that are
1. continuous in \((t, x)\),
2. locally Lipschitz in \(x\) uniformly with respect to \(t \in \mathbb{R}\),
3. bounded in \(t\) uniformly with respect to \(x\) in compact subsets of \(\mathbb{R}^n_0\).

An element \(X\) of \(\mathcal{X}(\mathbb{R}^n_0)\) is called a time-varying vectorfield. Clearly \(\mathcal{X}(\mathbb{R}^n_0)\) is closed under summation and multiplication with functions \(f \in \mathcal{F}(\mathbb{R}^n_0)\).

Associated to a time-varying vectorfield \(X \in \mathcal{X}(\mathbb{R}^n_0)\) is a differential equation
\[
\dot{x} = X(t, x)
\] (1)
on \(\mathbb{R}^n_0\) that has the existence and uniqueness property of trajectories. Its trajectory passing through state \(x_0\) at time \(t_0\) evaluated at time \(t\) is denoted by \(x(t, t_0, x_0)\). The map \((t, t_0, x_0) \mapsto x(t, t_0, x_0)\) is called the flow of this differential equation.

**Remark 1.** In general the flow \(x\) need not be forward complete: trajectories may escape to infinity or approach the origin in finite time. Similarly, the flow \(x\) may not be backward complete.

### 2.2 Push-forward

Let \(\phi : \mathbb{R}^n_0 \to \mathbb{R}^n_0\) be a smooth (that is, of class \(C^\infty\)) diffeomorphism. We are interested in “pushing forward” functions \(f \in \mathcal{F}(\mathbb{R}^n_0)\) and vectorfields \(X \in \mathcal{X}(\mathbb{R}^n_0)\) by this diffeomorphism.

The push-forward \(\phi_* f\) of \(f\) by \(\phi\) is defined as
\[
\phi_* f : \mathbb{R} \times \mathbb{R}^n_0 \to \mathbb{R} : (t, x) \mapsto (\phi_* f)(t, x) = f(t, \phi^{-1}(x)).
\] (2)

It is easy to see that \(\phi_* f\) is again in \(\mathcal{F}(\mathbb{R}^n_0)\).

For vectorfields, we first introduce the tangent map \(T_x \phi\) of \(\phi\) at \(x \in \mathbb{R}^n_0\):
\[
T_x \phi : \mathbb{R}^n \to \mathbb{R}^n : v \mapsto (T_x \phi)(v) = D\phi(x) \cdot v
\] (3)
where \(D\phi(x)\) is the Jacobian of \(\phi\) at \(x\) and where \(\cdot\) indicates the matrix product. The push-forward \(\phi_* X\) of \(X\) by \(\phi\) is then defined as
\[
\phi_* X : \mathbb{R} \times \mathbb{R}^n_0 \to \mathbb{R}^n : (t, x) \mapsto (\phi_* X)(t, x) = (T_{\phi^{-1}(x)} \phi)(X(t, \phi^{-1}(x))).
\] (4)

It is easy to see that \(\phi_* X\) is again in \(\mathcal{X}(\mathbb{R}^n_0)\).

### 2.3 Homogeneity

We give a geometric definition of homogeneity, in the spirit of [11, 3, 4].

Given \(r \in (\mathbb{R}_{>0})^n\), we introduce the 1-parameter family of dilations \(\delta_\lambda (\lambda > 0)\)
\[
\delta_\lambda : \mathbb{R}^n_0 \to \mathbb{R}^n_0 : x \mapsto \delta_\lambda(x) = (\lambda^r x_1, \ldots, \lambda^r x_n).
\] (5)
The orbits \(\Omega_x = \{\delta_\lambda(x) : \lambda > 0\}\) of this 1-parameter family of dilations are called homogeneous rays.

A time-varying vectorfield \(X \in \mathcal{X}(\mathbb{R}^n_0)\) is homogeneous of order \(\tau \in \mathbb{R}\) if
\[
(\delta_\lambda)_* X = \lambda^{-\tau} X \quad \forall \lambda.
\] (6)
A homogeneous norm is a smooth function \(\rho : \mathbb{R}^n_0 \to \mathbb{R}_{>0}\) that satisfies
\[
(\delta_\lambda)_* \rho = \lambda^{-1} \rho \quad \forall \lambda.
\] (7)

**Remark 2.** Equation (6) is equivalent to
\[
X(t, \delta_\lambda x) = \lambda^\tau \delta_\lambda X(t, x) \quad \forall \lambda, t, x.
\]
Equation (7) is equivalent to
\[
\rho(\delta_\lambda x) = \lambda \rho(x) \quad \forall \lambda, x.
\]

### 2.4 Stability and boundedness

**Definition 1 (local uniform asymptotic stability).** A differential equation \(\dot{x} = X(t, x)\) on \(\mathbb{R}^n_0\) with flow \(x(t, t_0, x_0)\) is locally uniformly asymptotically stable if

\[\text{S-1} \quad \forall c_2 > 0, \exists c_1 > 0 \text{ such that } \forall t_0, \forall x_0, \text{ if } \rho(x_0) < c_1 \text{ then } \rho(x(t, t_0, x_0)) < c_2 \forall t \geq t_0 \text{ in the domain of } x(\cdot, t_0, x_0)\]
and
\[\text{S-2} \quad \exists c_1 > 0 \text{ such that } \forall c_2 > 0, \exists T_0 \geq 0 \text{ such that } \forall t_0, \forall x_0, \text{ if } \rho(x_0) < c_1 \text{ then } \rho(x(t, t_0, x_0)) < c_2 \forall t \geq t_0 + T_0 \text{ in the domain of } x(\cdot, t_0, x_0)\]

**Definition 2 (boundedness).** A differential equation \(\dot{x} = X(t, x)\) on \(\mathbb{R}^n_0\) with flow \(x(t, t_0, x_0)\) is bounded if

\[\text{B-1} \quad \forall c_1 > 0, \exists c_2 > 0 \text{ such that } \forall t_0, \forall x_0, \text{ if } \rho(x_0) \leq c_1 \text{ then } \rho(x(t, t_0, x_0)) \leq c_2 \forall t \geq t_0 \text{ in the domain of } x(\cdot, t_0, x_0)\]
and
\[\text{B-2} \quad \exists c_2 > 0 \text{ such that } \forall c_1 > 0, \exists T_0 \geq 0 \text{ such that } \forall t_0, \forall x_0, \text{ if } \rho(x_0) \leq c_1 \text{ then } \rho(x(t, t_0, x_0)) \leq c_2 \forall t \geq t_0 + T_0 \text{ in the domain of } x(\cdot, t_0, x_0)\]

**Remark 3.** The notions of local uniform asymptotic stability and boundedness are defined in terms of a homogeneous norm \(\rho\). This is merely a matter of convenience: replacing the homogeneous norm \(\rho\) by the Euclidean norm would result in equivalent definitions.

**Remark 4.** The notion of local uniform asymptotic stability introduced in Definition 1 differs from the classical notion of local uniform asymptotic stability of an equilibrium point at the origin, in that the origin itself is excluded from the state-space.

**Remark 5.** Condition B-1 (resp. B-2) of Definition 2 corresponds to the notion of uniform boundedness (resp. uniform ultimate boundedness) from [12, 10] for differential equations on \(\mathbb{R}^n\).

---

1 We include the restriction “in the domain of \(x(\cdot, t_0, x_0)\)” because \(x(\cdot, t_0, x_0)\) may approach the origin in finite time and thus may not be forward complete. Similar remarks hold for conditions S-2, B-1 and B-2 introduced further in the paper.
3 Duality transformation

Associated to a 1-parameter family of dilations \( \delta_\lambda \) and a homogeneous norm \( \rho \), we introduce the smooth map

\[
S : \mathbb{R}_0^n \rightarrow \mathbb{R}_0^n : x \mapsto S(x) = \delta_{\rho(x)^{-2}}(x).
\]

(8)

Clearly \( S \) leaves homogeneous rays invariant, and on homogeneous rays, \( S \) is completely characterized by

\[
\rho(S(x)) = \rho(x)^{-1}.
\]

(9)

The map \( S \) may be interpreted as a reflection with respect to the homogeneous unit ball \( \{ \rho(x) = 1 \} \) along homogeneous rays.

It follows that \( S \circ S \) is the identity map on \( \mathbb{R}_0^n \), and thus, in particular, \( S \) is a smooth diffeomorphism. Furthermore

Lemma 1. \( S \) and the 1-parameter family of dilations \( \delta_\lambda \) are related by

\[
\delta_\lambda \circ S = S \circ \delta_{\lambda^{-1}} \quad \forall \lambda.
\]

(10)

We now introduce three \( \mathcal{F}(\mathbb{R}_0^n) \)-linear maps from \( \mathcal{X}(\mathbb{R}_0^n) \) onto \( \mathcal{X}(\mathbb{R}_0^n) \): \( X \mapsto X^S \), \( X \mapsto X^T \) and \( X \mapsto X^D \) defined by

\[
X^S = S \cdot X,
\]

(11)

\[
X^T(t, x) = -X(-t, x),
\]

(12)

\[
X^D = (X^S)^T.
\]

(13)

The two operations \( S \) and \( T \) commute and we may thus write

\[
X^D = (X^S)^T = (X^T)^S = X^{ST}.
\]

(14)

Furthermore

\[
X^{SS} = X^{TT} = X^{DD} = X.
\]

(15)

The vectorfield \( X^D \) is called the dual vectorfield associated to \( X \).

The trajectories of \( X \), \( X^S \), \( X^T \) and \( X^D \) are related:

Lemma 2. The following four statements are equivalent:

(i) \( t \mapsto \xi(t) \) is a solution of \( \dot{x} = X(t, x) \).

(ii) \( t \mapsto (S \circ \xi)(t) \) is a solution of \( \dot{x} = X^S(t, x) \).

(iii) \( t \mapsto \xi(-t) \) is a solution of \( \dot{x} = X^T(t, x) \).

(iv) \( t \mapsto (S \circ \xi)(-t) \) is a solution of \( \dot{x} = X^D(t, x) \).

The flows of \( X \) and \( X^D \), respectively denoted by \( x \) and \( x^D \), are thus related by:

\[
x(t_1, t_0, x_0) = x_1 \iff x^D(-t_0, -t_1, S(x_1)) = S(x_0).
\]

(16)

Duality Theorem 1 (stability and boundedness). The following two statements are equivalent:

(i) \( \dot{x} = X(t, x) \) is locally uniformly asymptotically stable.

(ii) \( \dot{x} = X^D(t, x) \) is bounded.

Schematically Theorem 1 may be represented as follows:

local uniform asymptotic stability \( \xrightarrow{D} \) boundedness

We emphasize that, although the duality transformation is based on a family of dilations and a homogeneous norm, the Duality Theorem 1 applies to general differential equations on \( \mathbb{R}_0^n \), not necessarily homogeneous.

Duality Theorem 2 (homogeneity). The following two statements are equivalent:

(i) \( X \) is homogeneous of order \( \tau \).

(ii) \( X^D \) is homogeneous of order \( -\tau \).

Schematically Theorem 2 may be represented as follows:

homogeneity of order \( \tau \) \( \xrightarrow{D} \) homogeneity of order \( -\tau \)

4 Local uniform asymptotic stability and boundedness of time-varying homogeneous differential equations

We first recall a stability and boundedness result for positive order homogeneous differential equations. Based on the duality introduced above, we then derive the dual result for negative order homogeneous differential equations.

Consider a time-varying vectorfield \( X \in \mathcal{X}(\mathbb{R}_0^n) \) that satisfies the following additional hypotheses:

H-1 \( X \) is periodic in \( t \) with period \( T > 0 \) independent of \( x \),

H-2 \( X \) is continuously differentiable in \( (t, x) \).

Various time-invariant vectorfields may be associated to \( X \): we introduce the time-averaged vectorfield \( X_{av} \)

\[
X_{av} : \mathbb{R}_0^n \rightarrow \mathbb{R}_0^n : x \mapsto X_{av}(x) = \frac{1}{T} \int_0^T X(t, x) \, dt,
\]

(17)

and a collection of time-frozen vectorfields \( X_\sigma (\sigma \in \mathbb{R}) \)

\[
X_\sigma : \mathbb{R}_0^n \rightarrow \mathbb{R}_0^n : x \mapsto X_\sigma(x) = X(\sigma, x).
\]

(18)

In general, a stability or a boundedness analysis of the time-varying differential equation

\[
\dot{x} = X(t, x)
\]

(19)

is highly nontrivial. Reducing the problem to the study of time-invariant differential equations

\[
\dot{x} = X_{av}(x)
\]

(20)

or

\[
\dot{x} = X_\sigma(x)
\]

(21)

may constitute an important simplification.

This has actually been proven to be possible for positive order homogeneous differential equations:
Theorem 3. Consider $X \in \mathcal{X}(\mathbb{R}^n_0)$ satisfying the hypotheses H-1 and H-2. Assume that $X$ is homogeneous of order $\tau > 0$. Then

1. the time-varying differential equation (19) is locally uniformly asymptotically stable if the time-averaged differential equation (20) is locally uniformly asymptotically stable,

2. the time-varying differential equation (19) is bounded if each time-frozen differential equation (21) is bounded.

Theorem 3 is essentially a paraphrased version of [9, Theorem 1] and [10, Theorem 3]. See also [6, Theorem 3]. The proof of this result is based on the inherently slow character of solutions of positive order homogeneous differential equations near the origin, and the inherently fast character far away from the origin.

We now illustrate the strength of the duality principle introduced in this paper. Starting from the result for positive order homogeneous differential equations Theorem 3, a straightforward application of the Duality Theorems 1 and 2 immediately yields the dual result for negative order homogeneous differential equations:

Corollary 1. Consider $X \in \mathcal{X}(\mathbb{R}^n_0)$ satisfying the hypotheses H-1 and H-2. Assume that $X$ is homogeneous of order $\tau < 0$. Then

1. the time-varying differential equation (19) is locally uniformly asymptotically stable if each time-frozen differential equation (20) is locally uniformly asymptotically stable.

2. the time-varying differential equation (19) is bounded if the time-averaged differential equation (20) is bounded.

Proof. First we prove part 1. Assume that each time-frozen differential equation $\dot{x} = X(\sigma)(x)$ is locally uniformly asymptotically stable. Then, by Theorem 1 each differential equation $\dot{x} = (X(\sigma))^{D}(x)$ is bounded. From (11), (12) and (13) it is immediate to see that $X(\sigma)^{D} = (X^{D})^{-\sigma}$. Hence each differential equation $\dot{x} = (X^{D})^{-\sigma}(x)$ is bounded. But $X^{D}$ is homogeneous of order $-\tau > 0$ by Theorem 2, and thus $\dot{x} = X^{D}(t, x)$ is also bounded by Theorem 3. Finally, by Theorem 1, we conclude that $\dot{x} = X(t, x)$ is locally uniformly asymptotically stable. The proof of part 2 is completely similar, and therefore left to the reader.

5 Conclusion

We have introduced a duality transformation between positive and negative order homogeneous vectorfields. We have then illustrated the strength of this duality concept: starting from a known stability and boundedness result for positive order homogeneous differential equations, we were able to obtain, without much effort, the dual result for negative order homogeneous differential equations.

Acknowledgments

This paper presents research results of the Belgian Programme on Interuniversity Poles of Attraction, initiated by the Belgian State, Prime Minister’s Office for Science, Technology and Culture. The scientific responsibility rests with its authors. The first author is supported by BOF grant 011D0696 of the Ghent University.

References
