On Design of Controllers for Linear Switched Systems with Guaranteed $H_2$-Type Cost

Izumi Masubuchi* and Makoto Tsutsui†

*Department of Systems and Computer Engineering, Kobe University
†Graduate School of Science and Technology, Kobe University
1-1 Rokkodai, Nada, Kobe 657-8501, JAPAN
{masubuti,makoto}@cs.kobe-u.ac.jp

Keywords: Switched systems, switched Lyapunov function, state jump, controller synthesis.

Abstract
Design of control systems with discrete switches of their dynamics is an important issue in control engineering and has been receiving much attention recently. In this paper, a controller synthesis approach is proposed for switched systems. We consider piecewise quadratic Lyapunov-like functions that leads to linear matrix inequalities (LMIs) for performance analysis and controller synthesis. State jumps of the controller responding to switches of plant dynamics are exploited to improve control performance.

1 Introduction
Lots of control systems work under discrete changes of their dynamics. They are caused by such as connections and disconnections of mechanical components, switches in electrical circuits, flow-control valves, and so forth. In these systems, which we call switched systems, discrete switches of their continuous dynamics often have great influence on their performance. Particularly discrete switches can not be ignored if the switching intervals are shorter than settling time of the continuous dynamics of the plant. Because of their increasing practical importance, switched systems have been receiving more and more attention recently. Particularly for stability analysis of switched systems has made great progress (See e.g. [6] and references therein).

This paper is concerned with synthesis of switched control systems excited with external switches that bring changes of dynamics and/or state jumps. Switching instance sequences are assumed to be measured on line and only a dwell time is available when a controller is synthesized. Based on a class of discontinuous Lyapunov-like functions, we show performance analysis of $H_2$-type cost for linear switched systems. In the synthesis problem to derive an output feedback switching controller with guaranteed $H_2$-type cost, we employ state jumps of the controller to improve stability and $H_2$-type control performance. The synthesis problem is reduced to solving LMIs as a generalization of LMI-based general synthesis method for linear time-invariant systems [7,8].

2 Analysis of linear switched systems

2.1 Linear switched systems
Let us consider a class of switched systems defined as follows. Denote by indices of $r$ distinguished modes of dynamics by $\mathcal{I} = \{1, 2, \ldots, r\}$ and a switching time sequence by $0 = t_0 < t_1 < t_2 < \cdots$. The mode activated in the interval $[t_k, t_{k+1})$ is represented by $\iota(k) \in \mathcal{I}$. If the number of switching is finite, formally we set $0 = t_0 < t_1 < \cdots t_k, t_{k+1} = t_{k+2} = \cdots = \infty$.

Assumption 1 1. There exists a dwell time constant $T_D > 0$ such that $t_{k+1} - t_k \geq T_D$ for all $k \geq 0$.

Let $x(t) \in \mathbb{R}^n$ be the system state and $x(t_0) = x_0$ be the initial value. For each $k$, the state $x(t)$ is continuous in $t \in [t_k, t_{k+1})$ and satisfies the following
Suppose that there exist $h(t)$ and $w(t)$, where $w(t)$ is a function of some adequate class. The initial value $x_{k+1}$ for each interval is given by

$$x_{k+1} = E_{h_i}x(t_{k+1}^{-}), \quad h = i(k + 1),$$

where $k \geq 0$ and $x(t_{k}^{-}) := \lim_{h \to t_{k}} x(t)$. The equation (2) represents possible state jumps of the system, which occur if $E_{h_i} \neq I$. In equations (1)–(2), the state $x(t) : t \in [0, \infty)$ is uniquely determined up to $\{i(k)\}_{k=0}^{\infty}$, $\{t_k\}_{k=0}^{\infty}$, $w(t)$ and $x_0$. The above class of switched systems is a linear version of systems considered in [1–5].

2.2 Stability analysis of linear switched systems with a dwell-time

In this subsection we show criteria for exponential stability of the linear switched systems, based on which $H_2$-type cost is considered in the next subsection.

**Definition 1** The origin of the system (1) is exponentially stable with stability degree $\beta$ if there exists a positive constant $M$ such that $\|x(t)\| \leq Me^{-\beta t}\|x(0)\|$.

Under the assumption that a dwell time is given, we utilize a class of Lyapunov-like functions [2] or weak Lyapunov functions [4]. Consider the following function defined in accordance with $\{t_k\}_{k=0}^{\infty}$ and $\{i(k)\}_{k=0}^{\infty}$:

$$v(x, t) = v_i(x), \quad t \in [t_k, t_{k+1}], i = i(k),$$

where each $v_i(x)$ is a $C^1$ function satisfying

$$a_i\|x\|^{2} \leq v_i(x) \leq b_i\|x\|^{2}.$$  

(4)

For any $x \in \mathbb{R}^n$ and $t \geq t_0$ we see that

$$a\|x\|^{2} \leq v(x, t) \leq b\|x\|,$$

$$a = \min_{i} a_i, \quad b = \max_{i} b_i.$$  

(5)

Obviously $v(x, t) \geq 0$, $x = 0 \Leftrightarrow v(x, t) = 0$, $\forall t$, and hence $v(x(t), t) \to 0$ as $t \to \infty$ implies $x(t) \to 0$.

**Proposition 1** Let $x(t)$ be the state of the system (1). Suppose that there exist $P_i$’s and $\alpha, \mu$ that satisfy $\alpha > 0$, $\mu > 0$, $\alpha T_D > \ln \mu$ and

$$\frac{d}{dt}v(x(t), t) < -2\alpha v(x(t), t), \quad t \in (t_k, t_{k+1})$$

(6)

$$v(x(t_k), t_k) < \mu^2v(x(t_k^+), t_k^+)$$

(7)

when $x(t) \neq 0$. Then the switched Lyapunov function $v(x(t), t)$ is monotonously decreasing in each interval $[t_k, t_{k+1})$, the sequence \{v(x(t_k), t_k)\} \infty_{k=0}^{\infty}$ is monotonously decreasing and the origin of the system (1) is exponentially stable with stability degree larger than $\beta := \min\{\alpha, \alpha - \ln \mu/T_D\} > 0$.

**Proof.** The inequality (6) implies that $v(x(t), t)$ is monotonously decreasing on $t \in [t_k, t_{k+1})$. From the Gronwall’s inequality and (6)–(7), it holds that

$$v(x(t_k), t_k) \leq \mu^2e^{-2\alpha(t_{k+1} - t_k)}v(x(t_k), t_k)$$

(8)

which yields

$$v(x(t_k), t_k) < \mu^2e^{-2\alpha(t_{k+1} - t_k)}v(x(t_0), t_0).$$

Thus we have

$$v(x(t), t) \leq e^{2\alpha(t-t_0)}v(x(t_0), t_0)$$

$$< \mu^2e^{-2\alpha(t-t_0)}v(x(t_0), t_0)$$

$$\leq \mu^2e^{-2\alpha(t-t_0)}v(x(t_0), t_0)$$

$$= e^{2\alpha(t-t_0)}v(x(t_0), t_0)$$

for $t \in [t_k, t_{k+1})$, $k \geq 0$, $t_k < \infty$, and

$$v(x(t), t) \leq e^{2\alpha(t-t_0)}v(x(t_0), t_0)$$

(9)

for $k = 0$. Setting $\beta = \min\{\alpha, \alpha - \ln \mu/T_D\}$, which is positive form the assumption, we obtain

$$\|x(t)\|^2 \leq \frac{1}{a}v(x(t), t) < \frac{1}{a}e^{-2\beta(t-t_0)}v(x(t_0), t_0)$$

$$\leq \frac{b}{a}e^{-2\beta(t-t_0)}v(x(t_0), t_0)\|x(t_0)\|^2$$

(10)

and hence $x(t)$ converges to zero exponentially.

We call the function (3) **switched Lyapunov function** if it satisfies the conditions in Proposition 1. A typical viewgraph of a switched Lyapunov function is shown in Fig.1.

Below we consider switched Lyapunov functions of piecewise quadratic form:

$$v_i(x) = x^TP_ix,$$

(8)

where $P_i$ is a positive definite symmetric matrix. Then $a = \min_i \lambda_{\min}(P_i)$ and $b = \max_i \lambda_{\max}(P_i)$, and above proposition is stated in terms of matrix inequalities for Lyapunov functions of piecewise quadratic form as follows.
Corollary 1 Suppose that there exist $P_i$’s and $\alpha, \mu$ that satisfy

$$\alpha > 0, \quad \mu > 0, \quad \alpha T_D > \ln \mu \quad (9)$$

$$P_i > 0, \quad i \in I \quad (10)$$

$$A_i^T P_i + P_i A_i + 2\alpha P_i < 0, \quad i \in I \quad (11)$$

$$\mu^2 P_i > E_{hi}^T P_h E_{hi}, \quad (h, i) \in S. \quad (12)$$

Then $v(x, t)$ defined in (8) is a switched Lyapunov function of the system (1).

2.3 $H_2$-type cost

Based on the switched Lyapunov function in the previous subsection, we consider performance analysis for an $H_2$-type cost of linear switched systems. Let $w(t)$ in system (1) be an impulse input and suppose that the initial state is set by the impulse input as $x(t_0) = Bw_0$ for some $w_0 \in \mathbb{R}^m$. Let $z(t)$ be the evaluated output as

$$z(t) = C_i x(t), \quad t \in [t_k, t_{k+1}), \quad i = i(k) \quad (13)$$

and consider the following $H_2$-type cost:

$$\gamma^* = \inf \gamma \text{ such that } \left( \int_0^\infty \|z(t)\|^2 dt \right)^\frac{1}{2} \leq \gamma \|w_0\|, \quad \forall w_0 \in \mathbb{R}^m \quad (14)$$

Theorem 1 Suppose that $P_i$’s, $\alpha, \mu$ and $\gamma$ satisfy $\alpha > 0, \quad \mu > 0, \quad \alpha T_D > \ln \mu$ and

$$P_i > 0, \quad i \in I, \quad (15)$$

$$A_i^T P_i + P_i A_i + \frac{1}{\gamma} C_i^T C_i + 2\alpha P_i < 0, \quad i \in I, \quad (16)$$

$$\gamma I > e^{2\alpha T_D} B_{hi}^T P_h B_{hi}, \quad (17)$$

$$\mu^2 P_i > E_{hi}^T P_h E_{hi}, \quad (h, i) \in S. \quad (18)$$

Then $x(t) \to 0$ exponentially and $\gamma^* < \gamma$.

Proof. Exponential convergence of $x(t)$ to zero is shown in Corollary 1. Below we evaluate the left side of (14). For each interval $(t_k, t_{k+1})$, $i = i(k)$, from (16)

$$\frac{1}{\gamma} \int_{t_k}^{t_{k+1}} \|z(t)\|^2 dt \leq \frac{1}{\gamma} \int_{t_k}^{t_{k+1}} x(t)^T C_i^T C_i x(t) dt$$

$$< \int_{t_k}^{t_{k+1}} -x(t)^T (A_i^T P_i + P_i A_i + 2\alpha P_i) x(t) dt$$

$$= \int_{t_k}^{t_{k+1}} \left\{ -\frac{d}{dt} v(x(t), t) - 2\alpha v(x(t), t) \right\} dt,$$

where $v(x(t), t) = v(x(t), t_0)$ is defined as (8). Denote $v_* = v(x(t_{k-1}^-), t_{k-1}^-)$. Since $v(t) \geq v_* e^{2\alpha(t_{k-1}^- - t)}$ holds for any $t \in [t_k, t_{k+1})$ from the Gronwall’s inequality, we see that

$$\int_{t_k}^{t_{k+1}} \left\{ -\frac{d}{dt} v(x(t), t) - 2\alpha v(x(t), t) \right\} dt$$

$$\leq v(x(t_k), t_k) - v_* - 2\alpha v_* \int_{t_k}^{t_{k+1}} e^{2\alpha(t_{k-1}^- - t)} dt$$

$$= v(x(t_k), t_k) - v_* + v_* (1 - e^{2\alpha(t_{k-1}^- - t)})$$

$$\leq v(x(t_k), t_k) - e^{2\alpha T_D} v(x(t_{k-1}^-), t_{k-1}^-). \quad (19)$$

On the other hand, (2) and (18) yields

$$\mu^2 v(x(t_{k-1}^-), t_{k-1}^-) = \mu^2 x(t_{k-1}^-)^T P_i x(t_{k-1}^-)$$

$$\geq x(t_{k-1}^-)^T E_{hi}^T P_h E_{hi} x(t_{k-1}^-) = v(x(t_{k-1}^-), t_{k-1}^-)$$

Applying this inequality to (19) with replacing $k + 1$ with $k$ and recalling $\mu^2 < e^{2\alpha T_D}$, we obtain

$$\mu^2 v(x(t_k^-), t_k^-) = \mu^2 x(t_k^-)^T P_i x(t_k^-)$$

$$< e^{2\alpha T_D} \{ v(x(t_k^-), t_k^-) - v(x(t_{k-1}^-), t_{k-1}^-) \},$$

where we assume $x(t_0^-) = x_0$.

Since $x(t)$ converges to zero exponentially, we have

$$\int_0^\infty \|z(t)\|^2 dt \leq \sum_{k=0}^\infty \int_{t_k}^{t_{k+1}} \|z(t)\|^2 dt$$

$$< \sum_{k=0}^\infty \gamma e^{2\alpha T_D} \{ v(x(t_k^-), t_k^-) - v(x(t_{k-1}^-), t_{k-1}^-) \}$$

$$= \gamma e^{2\alpha T_D} v(x_0)$$

$$= \gamma e^{2\alpha T_D} x_0^T P_0 x_0$$

$$= \gamma e^{2\alpha T_D} u_0^T B_{i_0}^T P_{i_0} B_{i_0} x_0$$

$$< \gamma^2 \|w_0\|^2,$$

where $i_0 = i(0)$, and the last inequality follows from (17). Thus the $H_2$-type cost $\gamma^*$ is proved to be less than $\gamma$. ■
3 Controller synthesis with guaranteed $H_2$-type cost

3.1 Description of control systems

Consider a feedback control system as in Fig. 2 under the external switching law. Let $\{t_k\}_{k=0}^{\infty}$ and $\{\iota(k)\}_{k=0}^{\infty}$ be the same as in the previous section. In addition, define $\mathcal{S} \subset \mathcal{I} \times \mathcal{I}$ as the set of pairs $(\iota(k+1), \iota(k))$ that can occur in a certain switching. Let $\iota(k) = i \in \mathcal{I}$ in $[t_k, t_{k+1})$ and represent the plant dynamics by

$$
\begin{bmatrix}
\dot{x}(t) \\
z(t) \\
y(t)
\end{bmatrix} =
\begin{bmatrix}
A_i & B_{i1} & B_{i2} \\
C_{i1} & 0 & D_{i12} \\
C_{i2} & D_{i21} & 0
\end{bmatrix}
\begin{bmatrix}
x(t) \\
w(t) \\
u(t)
\end{bmatrix},
$$

(20)

where $x(t) \in \mathbb{R}^n$ is the state, $w(t) \in \mathbb{R}^{m_1}$ is the external input, $u(t) \in \mathbb{R}^{m_2}$ is the control input, $z(t) \in \mathbb{R}^p$ is the controlled output and $y(t) \in \mathbb{R}^{p_2}$ is the measured output. The state jumps of the plant are represented by

$$
x(t_{k+1}) = E_{hi}x(t_{k+1}^{-}), \quad h = i(k+1).
$$

(21)

Remark 1 The above plant description includes either or both of jumps of the state and jumps of the coefficient matrices.

Consider the controller whose dynamics in $(t_k, t_{k+1})$ is given by

$$
\begin{bmatrix}
\dot{x}^c(t) \\
u(t)
\end{bmatrix} =
\begin{bmatrix}
A_i^c & B_i^c \\
C_i^c & 0
\end{bmatrix}
\begin{bmatrix}
x^c(t) \\
y(t)
\end{bmatrix},
$$

(22)

where $x^c(t) \in \mathbb{R}^n$ is the state of the controller with initial value $x^c(t_0) = x_0^c$. Denote the state jumps of the controller by

$$
x^c(t_{k+1}) = E_{hi}^c x^c(t_{k+1}^{-}), \quad h = i(k+1).
$$

(23)

The closed-loop system is represented as

$$
\begin{bmatrix}
\dot{x}^{cl}(t) \\
z(t) \\
x^{cl}(t_{k+1})
\end{bmatrix} =
\begin{bmatrix}
A_i^{cl} & B_i^{cl} \\
C_i^{cl} & 0 \\
E_{hi}^{cl}
\end{bmatrix}
\begin{bmatrix}
x^{cl}(t) \\
w(t)
\end{bmatrix},
$$

(24)

where $x^{cl} = [x^T \ (x^c)^T]^T$ is the state of the closed loop system and the coefficients are as follows:

$$
A_i^{cl} =
\begin{bmatrix}
A_i & B_{i2}C_i^c \\
B_i^c C_{i2} & A_i^c
\end{bmatrix},
$$

$$
B_i^{cl} =
\begin{bmatrix}
B_i \\
B_i^c D_{i21}
\end{bmatrix},
$$

$$
C_i^{cl} =
\begin{bmatrix}
C_{i1} & D_{i12} C_i^c
\end{bmatrix},
$$

$$
D_i^{cl} = D_{i11} + D_{i12} D_i^c D_{i21},
$$

$$
E_{hi}^{cl} =
\begin{bmatrix}
E_{hi} & 0 \\
0 & E_{hi}^c
\end{bmatrix}.
$$

The state of the closed loop system is unique up to $\{\iota(k)\}_{k=0}^{\infty}$, $\{t_k\}_{k=0}^{\infty}$, $w(t)$, $x_0$ and $x_0^c$.

Remark 2 Even though the state of the plant is continuous and never jumps, jumps of the controller’s state can bring better performance of the closed loop system. In the context of design of servosystems state jumps of the integrator are exploited for plants with a continuous state.

Now we formulate the synthesis problem.

Synthesis problem: Given the plant realization (20)–(21) and the dwell time $T_D$, find a controller (22)–(23) with which the closed loop system (24)–(25) satisfies the exponential stability and minimizes the $H_2$-type cost (14).
3.2 Synthesis via LMIs

In this section we show an LMI-approach to the design problem formulated in the above subsection. We will extend the general results of LMI-based controller synthesis to linear systems in [7, 8] for the synthesis of switched linear systems. The results of analysis in Section 2 give an upperbound for the optimal $H_2$-type cost with numerically tractable LMI conditions. From Theorem 1 a sufficient condition for $\gamma^* < \gamma$ is given by the following matrix inequalities of $\{P_i^c: i \in \mathcal{I}\}$, $K = \{A_i^c, B_i^c, C_i^c, D_i^c, E_{jk}; i \in \mathcal{I}, (j, k) \in \mathcal{S}\}$ and $\alpha, \mu$.

\[
P_i^c > 0, \quad i \in \mathcal{I}, \quad \alpha > 0, \quad \mu > 0, \quad \alpha T_D > \ln \mu,
\]

\[
Q_i := \begin{bmatrix} - (A_i^c)^T P_i^c - P_i^c A_i^c - 2\alpha P_i^c & (C_i^c)^T \end{bmatrix} > 0, \quad i \in \mathcal{I}, \tag{26}
\]

\[
R_i := \begin{bmatrix} \gamma e^{-2\alpha T_D} I & (B_i^c)^T P_i^c \\
(B_i^c)^T P_i^c & P_i^c \end{bmatrix} > 0, \quad i \in \mathcal{I}, \tag{27}
\]

\[
S_{hi} := \begin{bmatrix} \mu P_i^c & (E_{hi})^T P_{h_i} \\
E_{hi} P_{h_i} & \mu P_{h_i} \end{bmatrix} > 0, \quad (h, i) \in \mathcal{S}. \tag{28}
\]

It should be noted that this inequality is a bilinear matrix inequality of the variables $\{P_i^c\}_{i=1}^N$ and $K$ even if $\alpha$ and $\mu$ are fixed and still hardly tractable.

Below we will derive an LMI condition equivalent to the above inequality condition (26)–(29). We consider full order controllers (i.e., $n^c = n$). First, as in [8], without loss of generality $P_i^c$ is assumed to have the following structure:

\[
P_i^c = \begin{bmatrix} V_i & S_i \\ S_i & S_i \end{bmatrix}^{-1}, \quad V_i > S_i > 0.
\]

Setting a regular matrix $U_i$ as

\[
U_i = \begin{bmatrix} V_i & I \\ S_i & 0 \end{bmatrix},
\]

we define matrices:

\[
\hat{P}_i := U_i^T P_i^c U_i, \quad \hat{Q}_i := \begin{bmatrix} U_i^T \\ 0 \\ 0 \end{bmatrix} Q_i \begin{bmatrix} U_i & 0 \\ 0 & I \end{bmatrix},
\]

\[
\hat{R}_i := \begin{bmatrix} I & 0 \\ 0 & U_i^T \end{bmatrix} R_i \begin{bmatrix} I & 0 \\ 0 & U_i \end{bmatrix}, \quad \hat{S}_{hi} := U_i^T S_{hi} U_i.
\]

Then $\hat{P}_i, \hat{Q}_i, \hat{R}_i$ and $\hat{S}_{hi}$ are represented as the equalities in (31)–(33) with (34)–(38) by setting a new variable $p$ by

\[
p = \{V_i, W_i, F_i, G_i, H_i, L_i, Z_{jk}; i \in \mathcal{I}, (j, k) \in \mathcal{S}\},
\]

where

\[
W_i = (V_i - S_i)^{-1}, \quad F_i = \begin{bmatrix} I & 0 \\ W_i B_{i2} & -W_i \end{bmatrix},
\]

\[
D_i^c = \begin{bmatrix} C_i^c \\ B_i^c \end{bmatrix}, \quad \hat{A}_i = -A_i - A_i V_i S_i, \quad \hat{C}_i = \begin{bmatrix} I & C_2 V_i \\ 0 & S_i \end{bmatrix},
\]

$Z_{hi} = W_h (V_i - E_{hi} S_i)$,

and the BMI condition (26)–(29) is equivalent to the LMI condition (30)–(33):

\[
\hat{P}_i > 0, \quad i \in \mathcal{I}, \quad \alpha > 0, \quad \mu > 0, \quad \alpha T_D > \ln \mu, \tag{30}
\]

\[
\hat{Q}_i := \begin{bmatrix} - \hat{A}_i^T - \hat{A}_i - 2\alpha \hat{P}_i & \hat{C}_i^T \\
\hat{C}_i & \gamma I \end{bmatrix} > 0, \quad i \in \mathcal{I}, \tag{31}
\]

\[
\hat{R}_i := \begin{bmatrix} e^{-2\alpha T_D} I & \hat{B}_i^T \\
\hat{B}_i & \hat{P}_i \end{bmatrix} > 0, \quad i \in \mathcal{I}, \tag{32}
\]

\[
\hat{S}_{hi} := \begin{bmatrix} \mu \hat{P}_i & \hat{E}_{hi}^T \\
\hat{E}_{hi} & \mu \hat{P}_h \end{bmatrix} > 0, \quad (h, i) \in \mathcal{S}, \tag{33}
\]

where

\[
\hat{P}_i = \begin{bmatrix} V_i & I \\ I & W_i \end{bmatrix}, \quad \hat{A}_i = \begin{bmatrix} A_i V_i + B_{i2} F_i & A_i + B_{i2} H_i C_{i2} \\ L_i & W_i A_i + G_i C_{i2} \end{bmatrix}, \tag{34}
\]

\[
\hat{B}_i = \begin{bmatrix} B_{i1} + B_{i2} H_i D_{i21} \\ W_i B_{i1} + G_i D_{i21} \end{bmatrix}, \quad \hat{C}_i = \begin{bmatrix} C_{i1} V_i + D_{i12} F_i & C_{i1} + D_{i12} H_i C_{i2} \end{bmatrix}, \tag{35}
\]

\[
\hat{E}_{hi} = \begin{bmatrix} E_{hi} V_i & E_{hi} \\ Z_{hi} & W_h E_{hi} \end{bmatrix}. \tag{36}
\]

These are summarized in the following theorem.

Theorem 2 The inequalities (26)–(29) hold for some $\{P_i^c\}_{i=1}^N, K$ if and only if the LMIs (30)–(33) hold for some $p$. If the LMIs (30)–(33) has a solution $p$, one
of the solutions to the BMIs (26)–(29) is given by:

\[
\begin{bmatrix}
    P_i^1 \\
    P_i^2
\end{bmatrix} =
\begin{bmatrix}
    I & 0 \\
    B_{i2} & -W_i^{-1}
\end{bmatrix}
\begin{bmatrix}
    H_i & F_i \\
    G_i & W_i A_i V_i
\end{bmatrix}
\begin{bmatrix}
    I & C_{i2} V_i S_i^{-1} \\
    0 & S_i^{-1}
\end{bmatrix}
\]

where

\[E_{hi} = -W_i^{-1} (Z_{hi} - W_h V_i) S_i^{-1}\]

\[P_i^{cl} = \begin{bmatrix} V_i & S_i \\ S_i & S_i \end{bmatrix}^{-1}, \quad S_i = V_i - W_i^{-1}\]

Proof. The proof is similar to that of the main theorem of [7, 8].

4 Numerical examples

We present an illustrative numerical example for the proposed synthesis method. Set

\[\mathcal{I} = \{1, 2\}, \quad S = \{(1, 2), (2, 1)\}\]

and consider the following generalized plant with \(T_D = 1\):

\[
\begin{bmatrix}
    A_1 & B_{11} & B_{12} \\
    A_2 & B_{21} & B_{22}
\end{bmatrix} =
\begin{bmatrix}
    7 & 8 & 0 & 1 & 1 & -1 \\
    -3 & -3 & 1 & 0 & 1 & 1
\end{bmatrix},
\]

\[
\begin{bmatrix}
    C_{11} & D_{11} & D_{12} \\
    C_{21} & D_{21} & D_{22}
\end{bmatrix} =
\begin{bmatrix}
    1 & 0 & 0 & 0 & 0 & 0 \\
    0 & 1 & 0 & 0 & 0 & 0 \\
    0 & 0 & 0 & 0 & 0 & 0.1 \\
    1 & 1 & 0 & 1 & 1 & 0
\end{bmatrix},
\]

\[E_{21} = E_{12} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \]

Quadratic stabilization via uncommon controllers and a common Lyapunov function is infeasible for this system. Though for each of the two fixed systems an \(H_2\)-optimal (and stabilizing) controller is obtained, switching compensation with these individually synthesized controllers fails to stabilize the switched systems under switching, as shown in Fig.3.

Next, setting \(\alpha = 2\) and \(\mu = 7.352\), we minimized \(\gamma\) subject to (30)–(33) via a standard LMI solver LMI-Tools on Scilab. The derived switching controller is as follows:

\[A_i^c = \begin{bmatrix} 2275 & 2537 \\ -2273 & -2533 \end{bmatrix}, \quad B_i^c = \begin{bmatrix} -1.821 \\ 9.845 \end{bmatrix},
\]

\[C_i^c = \begin{bmatrix} -2272 & -2534 \\ -28.39 & 0.8521 \end{bmatrix}, \quad D_i^c = 0,
\]

\[A_i^c = \begin{bmatrix} -714.9 & 779.3 \\ -28.39 & 0.8521 \end{bmatrix}, \quad B_i^c = \begin{bmatrix} 2.348 \\ 13.05 \end{bmatrix},
\]

\[C_i^c = \begin{bmatrix} 705.4 & -775.2 \\ -28.39 & 0.8521 \end{bmatrix}, \quad D_i^c = 0,
\]

\[E_{21} = \begin{bmatrix} 1.004 & -0.027 \\ 0.024 & 0.863 \end{bmatrix}, \quad E_{12} = \begin{bmatrix} 1.649 & 0.7153 \\ -1.044 & -0.1413 \end{bmatrix}.
\]

Switched Lyapunov function \(v(x, t) = x^T P_i x\) is obtained as

\[
\begin{bmatrix}
    0.4241 & 0.3573 & -0.4241 & -0.3573 \\
    0.3573 & 0.3236 & -0.3573 & -0.3236 \\
    -0.4241 & -0.3573 & 0.6254 & 0.5736 \\
    -0.3573 & -0.3236 & 0.5736 & 0.5566 \\
\end{bmatrix},
\]

\[
\begin{bmatrix}
    0.9700 & -0.0959 & 0.9700 & 0.0959 \\
    -0.0959 & 0.1313 & 0.0959 & -0.1313 \\
    -0.9700 & 0.0959 & 0.9742 & -0.1006 \\
    0.0959 & -0.1313 & -0.1006 & 0.1536 \\
\end{bmatrix}.
\]

An upperbound of \(\gamma^*\) is derived as 1667.

The simulation results below are derived with the following switching sequence

\[\iota(k) = \begin{cases} 1, & k = 0, 2, 4, \ldots \\ 2, & k = 1, 3, 5, \ldots \end{cases}, \quad t_k = 0.5k\]

and the unit impulse input to the first element of \(w(t)\). Figure 4 shows the state \(x_{cl}(t)\) of the closed loop system and Fig.5 indicates the control input \(u(t)\). As in these figures the switching control works out well. The switched Lyapunov function \(v(x_{cl}(t), t)\) of the closed loop system is shown in Fig.6. The first five terms of \(v(x_{cl}(t_k), t_k)\) are 0.3236, 0.1063, 0.0008443, 0.0001962 and 0.0000015, with which we see that \(v(x_{cl}(t_k), t_k)\) monotonously decreases while \(v(x_{cl}(t), t)\) increases at switching instances.

5 Conclusions

In this paper we showed a performance analysis of \(H_2\)-type cost for linear switched systems. Standard LMI-based synthesis approach for linear time-invariant systems was generalized for synthesis of controllers with guaranteed \(H_2\)-type cost employing controlled jumps of the controller state.
References


